OXFORD IB DIPLOMA PROGRAMME

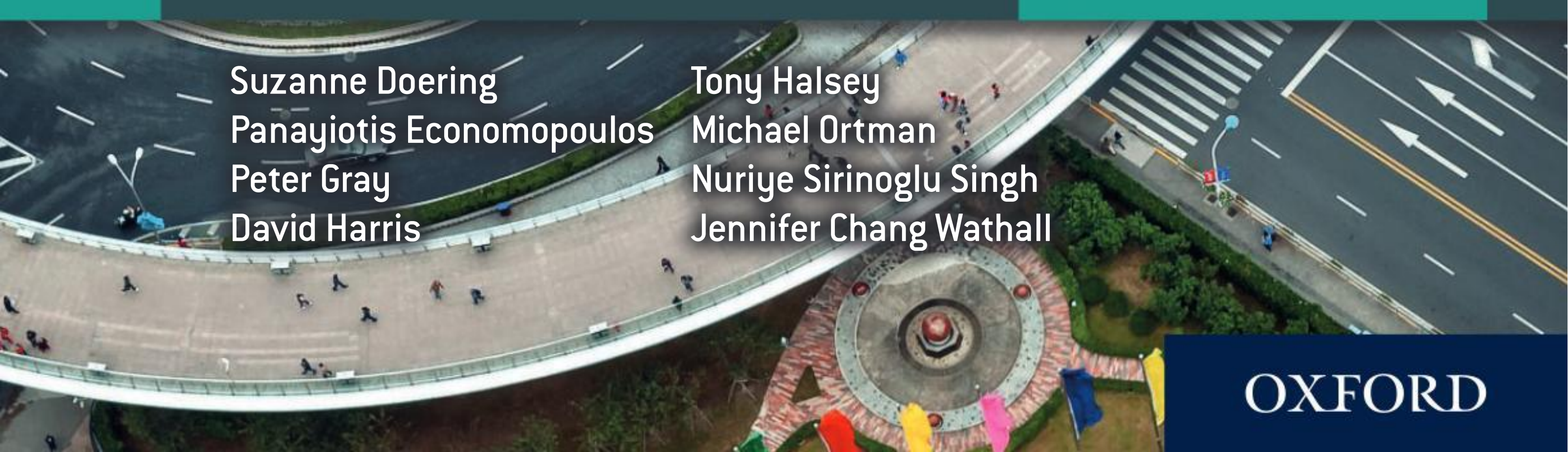
ADDITIONAL EXERCISES

MATHEMATICS:
APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE



Suzanne Doering
Panayiotis Economopoulos
Peter Gray
David Harris
Tony Halsey
Michael Ortman
Nuriye Sirinoglu Singh
Jennifer Chang Wathall

OXFORD

1.1 Representing numbers exactly and appproximately

1 Marketing data and reality

A company has decided to launch its first tablets.

Display	Camera	RAM	Operating system	Screen resolution	Battery	Battery life	Rate
7.9 in	8×10^6 pixels	3 GB	Android v8.1	1200 x 1920 pixels	6000 mAh	24 hours	4.5/5

a Discuss how precise these specifications are:

i State the number of significant figures for each specification and find the range of possible values accordingly.

ii Calculate the upper and lower bounds of each specification.

iii Calculate the percentage error of a tablet with the lower bound measurements.

2 Cosmetic factories use a machine to fill their bottles. Bottles of shampoo are to be filled to 500 ml (2 sf). A quality controller weighs empty and filled bottles. The scale he uses is correct to the nearest gramme, and the density of our shampoo is 2 g/ml. The following results were found:

Bottle	1	2	3
Empty bottle in g	11	9	10
Filled bottle in g	258	262	260

a **i** Find the quantity of shampoo in g and ml in each bottle

ii Determine whether the bottles should be accepted or rejected.

iii Calculate the average percentage error for each bottle.

iv If this 500 ml shampoo bottle costs $18 in a supermarket, find the range of prices of a litre of shampoo.

b Face creams are held in cylindrical containers with exterior radius 6.0 cm and height 1.5 cm (1dp).

i Calculate the range of volumes.

ii There is 50 ml of cream. If the inside cylinder is similar to the exterior one, find the measurements of the inside cylinder, rounding to 1 dp. Note that 1 ml is 1 cm^3.

iii Find the percentage error created by this rounding.

iv Calculate the actual range of volume if it is within 1% of the target of 50ml.

v State the rounding error it creates.

vi If such cream costs $12, find the range of prices for a litre of cream.

3 **a** Write the following quantities in standard notation:

i A light year is $9.5E^{15}$ metres.

ii The Milky Way has a diameter of 150,000 to 200,000 ly.

iii The diameter of the Milky Way in metres.

iv The maximum length of the lactose (milk) molecule is 0.0000000013349 m.

If we were to measure the Milky Way in terms of lactose molecules:

b find the minimum number of lactose molecules to draw a line across the Milky Way; write your answer in standard form rounded to 1 dp

c calculate the maximum percentage of the Milky Way that would fit into the diameter of a molecule of lactose; write your answer in standard form rounded to 2 dp.

Answers

1

	Display	Camera	RAM	O. S.	Screen Resolution	Battery	Battery life	Rate
	7.9"	8×10^6 $pixels$	3 GB	Android v8.1	1200 x 1920 pixels	6000 mAh	24 hours	4.5/5
i	2SF	1 SF	1 SF	NA	Ambiguous case: 3SF or 4SF	Ambiguous 2SF or 3SF or 4SF	2SF	2SF
ii	$7.95 \leq d < 8.05$	$4.5 \times 10^6 \leq c < 5.5 \times 10^6$	$2.5 \leq r < 3.5$	NA	$1195 \times 1915 \leq S_r < 1205 \times 1925$ or $1199.5 \times 1919.5 \leq S_r < 1200.5 \times 1920.5$	$4750 \leq b < 4850$ or $4795 \leq b < 4805$ or$4799.5 \leq b < 4800.5$	$23.5 \leq B_l < 24.5$	$4.45 \leq R < 4.55$
iii	0.62%	11.1%	20%	N.A	3SF: 0.42%×0.26% 4SF: 0.04%×0.03%	2SF:1.05% 3SF:0.1% 4SF:0.01%	2.13%	1.12%

2 a

	Bottle	1	2	3
i	Empty bottle in g	$10.5 \leq 11 < 11.5$	$8.5 \leq 9 < 9.5$	$9.5 \leq 10 < 10.5$
	Filled bottle in g	$257.5 \leq 258 < 258.5$	$261.5 \leq 262 < 262.5$	$259.5 \leq 260 < 260.5$
	Quantity of shampoo in g	$246 \leq w < 248$	$252 \leq w < 254$	$249 \leq w < 251$
	Quantity of shampoo in ml	$492 \leq c < 496$	$505 \leq c < 509$	$498 \leq c < 502$
ii	Accept/Reject	Reject: There is not enough product	reject :The bottle might leak	Accept
iii	% error	1.21%	1.19%	0%

iv 40$ / ml $\leq c$ 36$ /ml

b i 166.27 cm$^3 \leq V <$173.06 cm^3,

ii height=$\sqrt[3]{\frac{50}{16\pi}}$ so radius = 4 cm (1dp) and height = 1 cm (1dp),

iii 0.53%

iv 49.5ml $\leq v <$ 50.5ml

v rounded to 2 Sf or nearest unit

vi \$237.62≤ v <\$242.42

3 a i 9.5×10^{15}

ii 1.5×10^5 to 2×10^5

iii $1.5 \times 10^5 \times 9.5 \times 10^{15} = 1.425 \times 10^{21}$ to $2 \times 10^5 \times 9.5 \times 10^{15} = 1.9 \times 10^{21}$

iv 1.3×10^{-10}

b $\frac{1.425\times10^{21}}{1.3349\times10^{-9}} = 1.1 \times10^{30}$

c $\frac{1}{1.1\times10^{30}} = 1.05 \times10^{-35}$

1.2 Angles and triangles

1 China entered the Guinness Book of Records in May 2018 for flying 1374 drones simultaneously over the city of Xian.

All the following drones started directly over a clinometer O placed on the ground.

a D_1 flew north at a constant speed and altitude. At 5 seconds it was at an angle of elevation of $63°$ and 7 m north of the clinometer. Find:

i how far D_1 was from the clinometer, i.e. OD_1

ii how high D_1 was flying

iii the angle of elevation after flying five more seconds.

b D_2 flew north, too, but at a lower altitude than D_1. It reached a static position after 2 s. If from this static position it was 4.95 m away from D_1 at both 5 s and 10 s:

i calculate the angle between the two positions of D_1 from D_2 i.e. $\angle D_{1_5} D_2 D_{1_{10}}$

ii prove without using trigonometric ratios that D_2 flew at half the altitude of D_1.

c **i** D_3 flew vertically 12 m then 6 m at an angle of elevation of $30°$ East bound in 5 s. Calculate: a. its altitude; b. its horizontal distance from the clinometer.

ii D_3 then flew back at an angle of depression of $80°$ until it landed 5 s later. Calculate a. how far from the clinometer it landed; b. the distance between D_1 and D_3 at that time.

iii D_3 was supposed to land 1 m away from O, so was replaced by D_2. find: a. the angle of depression it should have taken; b. the distance D_2 flew to land 1 m East of O; c. the bearing D_2 flew

d D_4 flew west for 10 s, and its angle of elevation was then 20 degrees. If we assume it travelled at a constant speed and at the same altitude, calculate its angle of elevation after 5 s.

e D_5 flew up and could see marker M_1 on the ground at an angle of depression of $22°$ south. If it moved 500 m closer at the same altitude, the angle of depression to the object would be $41°$. Calculate:

i how high D_5 is over O

ii the distance to the object

iii the distance between O and the marker

iv the angle of depression to reach M_1 if D_5 flies 12 m south before descending

f D_6 needs to fly on a bearing of $334°$ for 13.2 m and then back for 7 m to a marker M_2 situated north of O.

i Show that there is more than one possible position for the marker.

ii State how far from O the marker can be situated.

g D_7 lost connection for a few seconds. It then appeared north of M_3 and at a bearing of $295°$ from M_4, two markers placed 10 m away. M_3 and M_4 were situated at an angle of depreciation of $15°$ and $25°$, respectively. Calculate how high D_7 was flying then.

2. The Spanish Architect Antoni Gaudi decorated the wall of the Park Guel in Barcelona using a mosaic of broken tiles. Discuss whether the area of the following broken tiles is enough to create a Gaudi style 1.3 × 1.3 m table.

a ΔABC, with $AC = 13\ cm,\ AB = 21\ cm\ ,\ \angle A = 59°$

b ΔABC, with $AB = 0.21\ m,\ BC = 0.31\ m\ , \angle A = 121°$

c ΔABC, with $AC = 80\ mm,\ BC = 90\ mm\ , \angle A = 12°$

d ΔABC, with $AC = 13\ cm,\ BC = 6cm\ , \angle A = 9\ °$

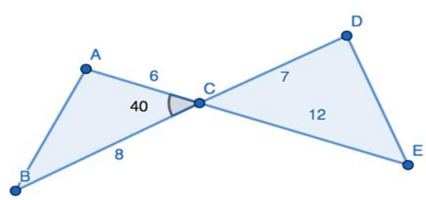
e the quadrilateral $ABCD$, with $AB = AD = 20c\ m,\ BC = CD\ , \angle A = 25°,\ \angle C = 40°$

f the quadrilateral $ABCD$, with $AB = 10cm,\ AC = 15,\ AD = 20cm,\ \angle BAD = 90°,\ \angle CAD = 34°$

g the quadrilateral $ABCD$, with $AB = 8\ cm,\ DC = 12\ cm\ , AD\ is\ 2cm$ longer than $BC\ \angle A = 50°,\ \angle C = 60°$

h a regular hexagon of side 20 cm with a regular pentagonal hole with side 10 cm.

i



3 A toy game is made of gears placed next to each other. The smallest gear has a radius of 10 cm and has 12 teeth, the second has a radius of 11 cm.

a Calculate how far are the teeth from each other.

b The first gear rotates at $60°$. Find:

i the arc length to which this movement corresponds

ii the angles at which the second gear will thus rotate

iii the radius of the 3rd gear, if it rotates at $30°$.

c If the first gear has radius r and rotates at $\theta°$ per second and each gear is enlarged by factor of a:

i write down the arc length corresponding to $\theta°$ on the first gear as an equation in terms of θ, r and a

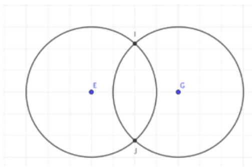
ii show that the speed generated on the 2nd gear is $\alpha = \theta a$ degrees per second

iii find an expression for the speed at the 3rd gear.

iv Hence find the speed of the nth gear.

4 Two light projectors are placed 3 m directly above the stage, 4 m away from each other. Their diameter is 20 cm and the angle they make with the vertical is $45°$. Calculate:

a The diameter of the circle of light a projector makes on the floor



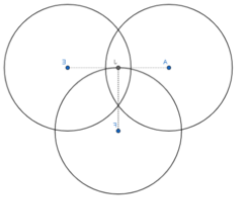
b $J\widehat{G}E$

c the area of the sector IGJ

d the area of ΔIGJ

e hence the overlapping area.

f A third projector is placed at the front such that its circle of light is tangential to the mid-point of the centre of the first two. Calculate:



i the area of triangle made by the centre of the three circles

ii the total area of light created by those three projectors.

Answers

1 **a** **i** 15.42 m

ii 13.74 m

iii $44.46°$

b **i** $90°$

ii $\angle D_{1_5}D_2D_{1_{10}}$=90° so $\angle D_{1_5}D_{1_{10}}D_2$=$\angle D_2D_{1_5}D_{1_{10}}$=45°.

As $D_{1_5}D_{1_{10}}//F_{1_5}F_{1_{10}}$ with $F_{1_5}F_{1_{10}}$the vertical projection of $D_{1_5}D_{1_{10}}$ onto the floor or altitude 0,

therefore $\angle D_{1_5}D_{1_{10}}D_2$=$\angle D_2F_{1_5}F_{1_{10}}$ = $\angle F_{1_5}F_{1_{10}}D_2$=$\angle D_2D_{1_5}D_{1_{10}}$=45° alternate angles

As D_1's altitude is constant $D_{1_5}D_{1_{10}} \perp D_{1_5}F_{1_5}$ so $D_{1_5}D_{1_{10}}F_{1_{10}}F_{1_5}$ and D_2 is the point of intersection of the diagonal hence it is altitude is half D_1's altitude

c **i** a. $15\ m$; b. $5.2\ m$

ii a. $2.56\ m$; b. 14.31 m

iii a. $74.36°$; b. 12.59 m; c. $176.56°$

d Let x be the horizontal distance travelled by the drone, h be the height, θ the angle of elevation $\frac{h}{x} = \tan 20$ and $\frac{h}{2x} = \tan\theta$ then $\tan 20 = 2\tan\theta$. $\theta = 10.31$

e **i** 377.44 m

ii 1007.56 m

iii 434.19 m

iv $\tan^{-1}\frac{389.56}{44.66-12} = 85.2\text{m}$

f **i** Let A be the position at which the drone turns and M be the position of the marker. $\angle OMC = \sin^{-1}\left(\frac{13.2\times\sin 26}{7}\right)$= $55.75°$ or $124.25°$ there is thus two positions for the markers.

ii 15.8 m or 7.9 m

g 2.9 m

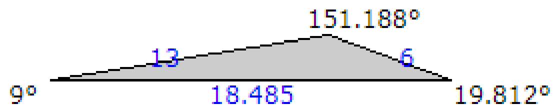
2

a $0.0117\ \text{m}^2$

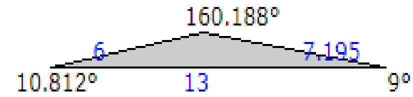
b $0.0129.8\ \text{m}^2$

c $0.00139\ \text{m}^2$

d 0.00188m^2 or 0.000732m^2 e.g



or



e v.$0.013603\ \mathrm{m}^2$

f $0.014606\ \mathrm{m}^2$

g $0.014572\ \mathrm{m}^2$

h $0.08672\ \mathrm{m}^2$

i $0.004243\ \mathrm{m}^2$

The total area is thus either $1.8997\ \mathrm{m}^2$ or $1.89858\ \mathrm{m}^2$. In both cases there should be enough tiles to cover the whole table. The pieces, however, have to be broken up into pieces to fit in.

3 **a** $\frac{2\pi\times10}{15} = 4.19$ cm

b **i** $\frac{60\times2\pi\times10}{360} = 10.47\ \text{cm}$

ii $\frac{\theta\times2\pi\times11}{360} = 10.47\ \text{ cm } \theta = 54.54°$

iii $\frac{30\times2\pi\times r}{360} = 10.47\ \text{ cm}, r = 20\ \text{cm}$

c **i** $\frac{\theta\times2\pi\times r}{360}$

ii the radius of the second gear is $\frac{r}{a}$ so $\frac{\theta\times2\pi\times r}{360} = \frac{\alpha\times2\pi\times\frac{r}{a}}{360}$ so $\alpha = \theta a$

iii the radius of the third gear is $\frac{1}{a}\times\frac{r}{a}$ so $\frac{\theta\times\pi\times r^2}{360} = \frac{\alpha\times2\pi\times\frac{r}{a^2}}{360}$ so $\alpha = \theta a^2$ ° per second

iv θa^{n-1} ° per second

4 **a** $0.02 + 2(3\tan 45)$=6.02m

b Let M the midpoint between E and G EG=4, ΔMGJ is a right angle triangle, MG=2, IG=6.02 so $0.02 + 2(3\tan 45)$=6.02m $J\widehat{G}E = \cos^{-1}\frac{2}{6.02} = 70.6°$

c $\frac{141.2\times\pi\times6.02^2}{360} = 44.66\ \mathrm{m}^2$

d the area of ΔIGJ is twice ΔMGJ $MJ = \sqrt{GI^2 - MG^2} = 5.68$

so area= $5.68\times2 = 11.36\ \mathrm{m}^2$

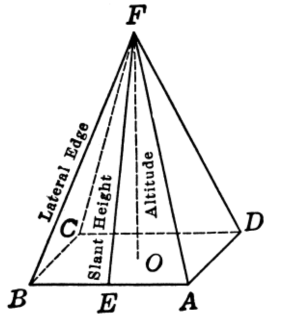
e $2\times(44.66 - 11.36) = 66.6\mathrm{m}^2$

f **i** 12.04m^2

ii $3\pi\times6.02^2 - 12.04 - 3.33 - (20.585\times2) = 255.04$ m^2

1.3 Three-dimensional geometry

An indoor playground consists of a rope Jungle Gym.



a The first attraction is regular square-based pyramids (all edges in equal in length) with side 7 m. Calculate:

i $\angle FAB$

ii the height of the pyramid

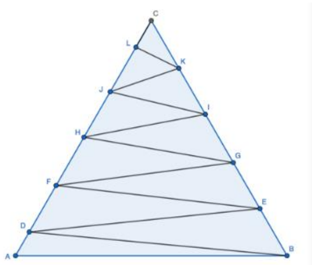
iii hence, its volume

iv $\angle FEO$, $\angle FBO$ and $\angle CFA$

v the length of the rope at the slant FE

vi hence, find the lateral area of the pyramid.

b From each vertex A, B, C and D, 2 oblique ropes pass through regular points on the consecutive lateral edges in both direction. Let R_{Bc} be the rope starting at B, passing through the points $Z, Y, X \dots F$ on AF, DF, $CF\dots$, respectively, such that $AZ = \frac{1}{n}AF$, $DY = \frac{2}{n}DF$, $CX = \frac{3}{n}CF$. If n=4, find:



i the length of one rope

ii the angle the ropes make on the lateral faces whenever they intersect

iii the two internal angles the ropes make whenever they intersect at Y

iv the angle a rope makes each time it meets a lateral edge.

c For any n, show that the formula for:

i the total length of a rope is $\frac{7}{n}\sum_{n=1}^{i}\sqrt{i^2+1}$

ii the angle $\angle K$ the ropes make on the lateral faces whenever they intersect at AK=$\frac{k}{n}AF$ is $\cos^{-1}\left(\frac{(n-k)^2-1}{\left(\sqrt{((n-k)^4+(n-k)^2+1)}\right)}\right)$

iii the angle $\angle K$ a rope makes each time it meets a lateral edge is $90°$

iv Verify your results and justify any discrepancy.

d For a pyramid where BA = a meters and FA = ka meters with $k, a \in \mathbb{R}$. Find:

i the general formula for the height of the pyramid in terms of k and a

ii the general formula for the volume

iii the slant FE

iv the general formula for the lateral area

e Another installation is made of four truncated 2 m high cylinders with radius 1 m and chord length of 1 and a cuboid in the middle. Find:

i its volume

ii its surface area. (Ignore the bottom part as it lies on the floor).

iii the amount of rope needed to make the perimeter of the truncated section.

iv the longest diagonal rope.

f The last attraction is an $12\pi\ m^3$ cone with surface area $24\pi\ m^3$ with an interior dodecagonal prism.

i calculate: a) the radius; b) the height; c) the slant; d) the angle at the vertex of the cone

ii show that the maximum volume of the interior dodecagonal prism is V = $12x^2 - 4x^3$.

Answers

a **i** As all edges are equals, the lateral triangles are equilateral so $\angle FAB = 60°$

ii BD=$\sqrt{\mathbf{7^2 + 7^2}} = 7\sqrt{\mathbf{2}}$ so DO=$\frac{\mathbf{1}}{\mathbf{2}}7\sqrt{\mathbf{2}}$, and FO=$\sqrt{\mathbf{7^2} - \left(7\frac{\sqrt{2}}{2}\right)^{\mathbf{2}}} = 7\sqrt{\frac{\mathbf{1}}{\mathbf{2}}}$=4.95m

iii $V = \frac{\mathbf{1}}{\mathbf{3}} \times \mathbf{7^2} \times 7\sqrt{\frac{\mathbf{1}}{\mathbf{2}}} = \frac{\mathbf{7^3}}{\mathbf{3\sqrt{2}}}$=$80.85m^3$

iv $\angle FEO = \tan^{-1}\left(\frac{FO}{OD}\right) = 54.74$, $\angle FBO = \tan^{-1}\left(\frac{FO}{OB}\right) = 45°$ $\angle AFC$=$90°$

v FE= $7\sqrt{\frac{\mathbf{3}}{\mathbf{4}}}$ =6.06m

vi $4 \times 7\sqrt{\frac{\mathbf{3}}{\mathbf{4}}} \times \mathbf{7} \times \frac{\mathbf{1}}{\mathbf{2}}$=$7^2\sqrt{\mathbf{3}}$= 84.87m^2

b **i** $XF = 1.75\,m$

$\angle YFX = 60°$, $YF = 3.5$ using cosine rule $XY = 3.03\,m$

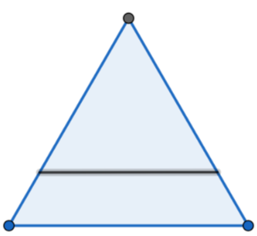
Using sine rule we get $\angle FXY = 90$, hence $ZY = 4.63m$

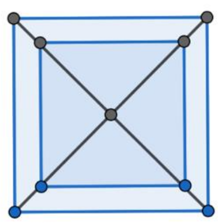
BA = 7 m, $\angle FAB = 60°$, $AZ = \frac{7}{4}$ m using cosine rule $BZ = 6.31\,m$

So the total garland is 15.72 m long.

ii angles on the lateral face $\angle X = 90°$, $\angle Y =$ 49°, $\angle Z = 33°$

iii





$Z_1\ Z_2 = \sqrt{2\left(7 \times \frac{3}{4}\right)^2}$=$\frac{21}{4}\sqrt{2}$

As $ZY = 4.62m$

$$\angle Z_1YZ_2 = \cos^{-1}\left(\frac{4.63^2 + 4.63^2 - \left(\frac{21}{4}\sqrt{2}\right)^2}{2 \times 4.63^2}\right) = 107°$$

Similarly $X_1\ X_2 = \sqrt{2\left(7 \times \frac{1}{4}\right)^2}$=$\frac{7}{4}\sqrt{2}$

As $XY = 3.01m$

$$\angle X_1YX_2 = \cos^{-1}\left(\frac{3.03^2 + 3.03^2 - \left(\frac{7}{4}\sqrt{2}\right)^2}{2 \times 3.01^2}\right) = 49°$$

iv $\angle AFC = 90°$,BY=$\sqrt{7^2 + \left(7 \times \frac{1}{2}\right)^2} = 8.75$, $\angle BZY = \cos^{-1}\left(\frac{6.31^2 + 4.63^2 - (8.75)^2}{2 \times 6.3 \times 4.63}\right) = 90°$

ZX=$\sqrt{\left(\frac{7}{4}\right)^2 + \left(7 \times \frac{3}{4}\right)^2} = 30.625$, $\angle ZYX = \cos^{-1}\left(\frac{3.03^2 + 4.62^2 - (30.625)^2}{2 \times 3.03 \times 4.63}\right) = 90°$

c **i** $\sum_{n=1}^{i}\sqrt{\left(\frac{i}{n}\right)^2+\left(\frac{i+1}{n}\right)^2-2\times\frac{i}{n}\times\frac{i+1}{n}\times cos60}=\sum_{n=1}^{i}\sqrt{\left(\frac{i}{n}\right)^2+\left(\frac{i+1}{n}\right)^2-\frac{i^2+i}{n^2}}=$

$\sum_{n=1}^{i}\sqrt{\left(\frac{i}{n}\right)^2(i^2+1)}=\frac{1}{n}\sum_{n=1}^{i}\sqrt{i^2+1}$

ii let the angle be. $\angle IKJ$,

$\mathrm{IK}^2=\left(\frac{n-(k+1)}{n}\right)^2+\left(\frac{n-k}{n}\right)^2-2\left(\frac{n-(k+1)}{n}\right)\left(\frac{n-k}{n}\right)\cos 60$

$=\frac{1}{n^2}((n-k)^2-2(n-k)+1+(n-k)^2-(n-k)^2+(n-k))$

So, IK$=\frac{1}{n}\sqrt{(n-k)^2-(n-k)+1}$

Similarly , JK$=\sqrt{\left(\frac{n-(k-1)}{n}\right)^2+\left(\frac{n-k}{n}\right)^2-2\left(\frac{n-(k-1)}{n}\right)\left(\frac{n-k}{n}\right)\cos 60}=\frac{1}{n}\sqrt{(n-k)^2+(n-k)+1}$

As $\angle AFC=90°$

$\mathrm{IJ}=\frac{2}{n}$

$$\angle IKJ=\cos^{-1}\left(\frac{\frac{(n-k)^2+(n-k)+1+(n-k)^2-(n-k)+1-4}{n^2}}{\frac{2\left(\sqrt{(n-k)^2+(n-k)+1}\right)\left(\sqrt{(n-k)^2-(n-k)+1}\right)}{n^2}}\right)$$

$$=\cos^{-1}\left(\frac{(n-k)^2-1}{\left(\sqrt{((n-k)^2+1)^2-(n-k)^2}\right)}\right)$$

$$=\cos^{-1}\left(\frac{(n-k)^2-1}{\left(\sqrt{((n-k)^4+(n-k)^2+1)}\right)}\right)$$

iii As $\angle AFC=90°$, $\mathrm{IJ}^2=\left(\frac{n-(k+1)}{n}\right)^2+\left(\frac{n-(k-1)}{n}\right)^2=\frac{2(n-k)^2+2}{n^2}$ and as

IK$=\frac{1}{n}\sqrt{(n-k)^2+(n-k)+1}$, JK$=\frac{1}{n}\sqrt{(n-k)^2-(n-k)+1}$

$$\angle IKJ=\cos^{-1}\left(\frac{\frac{(n-k)^2+(n-k)+1+(n-k)^2-(n-k)+1-(2(n-k)^2+2)}{n^2}}{\frac{2\left(\sqrt{(n-k)^2+(n-k)+1}\right)\left(\sqrt{(n-k)^2-(n-k)+1}\right)}{n^2}}\right)=\cos^{-1}(0)=90=$$

d **i** BD$=\sqrt{a^2+a^2}=a\sqrt{2}$ so DO$=\frac{a\sqrt{2}}{2}=\frac{a}{\sqrt{2}}$, and FO$=\sqrt{(ka)^2-\left(\frac{a}{\sqrt{2}}\right)^2}=a\sqrt{k^2-\frac{1}{2}}$

ii $\frac{a^3}{3}\sqrt{k^2-\frac{1}{2}}$

iii $FE=\sqrt{a^2\left(k^2-\frac{1}{2}\right)+\left(\frac{a}{2}\right)^2}=a\sqrt{k^2-\frac{1}{4}}$

iv $4\times\frac{a^2}{2}\sqrt{k^2-\frac{1}{4}}=2a^2\sqrt{k^2-\frac{1}{4}}$

e **i** as the chord is 1 m the angle of the chord is 60 and the cuboid has a square face of 1 m by 1 m

$2\left(4\times\left(\frac{300}{360}\pi\times1^2+\frac{1}{2}\times1^2\times\sin(60)\right)+1^2\right)$ =26.4 m^3

ii $4\times\left(\frac{300}{360}\pi\times1^2+\frac{1}{2}\times1^2\times\sin(60)\right)+1^2$ +2$\left(4\times\left(\frac{300}{360}2\pi\times1\right)\right)$=55.1 m^2

iii $\frac{-300}{360}\times2\times\pi\times1=1\frac{2}{3}\ \pi$

iv max horizontal distance = $1+2\times\frac{\sqrt{3}}{2}=1+\sqrt{3}$, max vertical distance = 2

diagonal= $\left(1+\sqrt{3}\right)^2+4=11.46\,m$

f **i** r = 3, h = 4, s = 5

ii a dodecagon is made of twelve isoceles triangles. Let x be the equal side with $0<x\leq3$

area of cross section= $12\times\frac{1}{2}\times x^2\times sin30$=$3x^2$

using similar triangle $\frac{h}{3-x}=\frac{4}{3}$ so $h=4-\frac{4}{3}x$

so, V = $(4-\frac{4}{3}x)$ $3x^2$=$12x^2-4x^3$.

2.1 Collecting and organising data

1 State which of the following are qualitative, quantitative, discrete or continuous.

a The number of ducks in a pond.

b The water is blue.

c The time taken for 10 dogs to eat a bowl of food.

d The number of petals on tulips.

e The children are happy.

f The length of 5-year-olds' arm spans.

g The height of trees in a forest.

h Jean sizes.

2 Engy works in a windmill that is open to the public one day each year. Engy measures the time, to the nearest tenth of a minute, that it took each of the 250 visitors to climb to the top of the windmill.

14.4 08.9 04.7 12.8 14.2 06.5 06.8 08.4 08.7 09.6 04.6 15.8 06.2 17.8 19.8 04.6 16.1 08.7 08.7
09.0 20.0 06.1 12.8 12.6 03.4 03.3 16.1 04.1 10.8 03.0 11.7 19.4 19.1 06.0 17.0 02.7 20.4 04.3
19.6 06.4 16.7 16.8 20.4 05.6 18.6 16.7 07.4 14.6 12.6 14.0 07.8 03.3 06.0 11.1 15.9 04.9 02.9
19.3 16.8 04.2 04.0 06.2 16.6 16.8 09.3 13.9 20.0 06.2 09.0 03.8 13.7 09.4 14.4 03.5 09.1 06.8
20.0 16.2 16.4 20.4 07.8 09.5 03.9 11.5 10.7 20.3 11.0 07.2 14.6 03.6 05.9 06.6 13.1 03.3 07.3
05.6 11.3 09.9 07.5 05.3 15.5 10.0 10.6 08.7 09.4 02.9 20.2 20.0 08.6 02.9 12.8 13.2 09.1 09.8
09.1 15.6 13.9 20.3 19.8 10.6 08.8 07.9 14.1 12.1 17.9 18.1 05.3 14.3 13.9 08.6 10.2 19.2 19.5
09.8 13.2 14.0 09.3 11.0 10.0 08.6 19.9 19.3 17.9 03.1 05.6 10.6 17.0 16.2 13.7 16.4 07.7 06.4
11.7 09.4 16.5 02.6 16.7 18.7 13.6 15.7 06.1 06.0 03.1 11.3 19.4 08.2 05.9 16.8 03.0 05.0 09.4
10.4 14.9 14.7 16.0 18.4 04.8 08.1 02.7 13.8 08.4 13.7 15.1 04.6 12.1 04.2 06.6 15.8 07.5 10.2
16.3 19.5 14.1 16.4 17.1 19.2 13.2 10.5 10.1 08.3 16.7 18.1 11.9 07.3 10.5 13.0 10.1 13.2 14.2
16.6 15.5 09.0 14.9 17.6 07.5 15.1 17.7 06.7 15.3 15.2 07.4 16.6 16.5 17.1 08.3 17.4 08.5 06.5
18.0 03.3 10.0 16.9 20.7 09.1 11.2 18.6 05.0 19.4 03.2 16.1 04.2 13.6 11.8 02.4 12.9 14.7 06.5
10.4 17.4 13.0

a Take a systematic sample of every 10^{th} time and find the mean and standard deviation.

b Use a random number generator to pick 25 numbers between 1 and 250 and find the mean and standard deviation of these times.

c Given that the mean of all 250 times is 11.5 and the standard deviation is 5.19, comment on your two sets of results. The 250 people consisted of 100 men, 60 women and 90 children. Engy wants to take a stratified sample of 30 from the 250.

d Work out how many from each group she should pick.

Answers

1 **a** Quantitative and discrete

b Qualitative

c Quantitative and continuous

d Quantitative and discrete

e Qualitative

f Quantitative and continuous

g Quantitative and continuous

h Quantitative and discrete

2 **a** Students' own answers.

b The answer will depend on which numbers the random generator produced.

c This answer will depend on the results that the students found in parts (i) and (ii).

d $\frac{100 \times 30}{250} = 12 \text{ men}, \frac{60 \times 30}{250} = 7.2 \text{ women}, \frac{90 \times 30}{250} = 10.8 \text{ children}.$ So, she needs to pick at random 12 men, 7 women and 11 children.

2.2 Statistical measures

1 A creche has 22 children from the ages of 1 to 4 years. The children's weights, in kilograms, are:

9.4	10.3	11.6	9.7	12.8	10.4	12.6	14.3	24.8	10.6	8.8
13.5	14.1	11.5	12.1	13.7	12.8	11.9	9.3	10.6	11.3	9.9

a Find the mode, median and mean.

b Find the standard deviation and interquartile range.

c Find if there are any outliers.

d Find the mean and standard deviation without the outlier(s) and comment on your answers.

2 Yeliena has 36 hens. She notes how many eggs each hen lays in one week

Number of eggs	Frequency
4	4
5	8
6	16
7	6
8	2

a Write down the mode.

b Find the mean and standard deviation.

c Find the median and interquartile range.

3 Marit was testing the reaction times, t seconds, of 25 people in his class.

Time, t seconds	Frequency
$0 < t \le 1$	2
$1 < t \le 2$	12

$2 < t \leq 3$	x
$3 < t \leq 4$	2
$4 < t \leq 5$	3
$5 < t \leq 6$	1

a Find the value for x.

b Write down the modal class.

c Write down the mid-point of the modal class.

d Find an estimate of the mean and standard deviation.

e Explain why these are only estimates.

4 Mr Lu marks a Chinese test out of 45. The mean mark was 32 with a standard deviation of 4. He has to change the marks into percentages for the reports. He has three choices:

A Multiply each mark by 2 and add 10.

B Multiply each mark by 2.2 and add 1.

C Multiply each mark by 2.4 and subtract 8

a Find the new mean and standard deviation for each choice. Simon has 40 / 45 and Millie has 20 / 45.

b Find their percentage mark for each of the three choices.

c Which choice do you think they will prefer? Give a reason for your answer.

5 In Mr Ali's boxing club there are 16 members. The mean number of push-ups they can do is 68. Erhan joins the group and the mean number of push-ups is now 70. Find how many push-ups Erhan can do.

6 The mean height of 16-year-old boys is 1.53 m and standard deviation is 0.1 m. The mean height of 16-year-old girls is 1.56 m and standard deviation is 0.08 m. Find the mean height and standard deviation in a class of 10 boys and 15 girls.

Answers

1 **a** Modes are 12.8 and 10.6; median = 11.55; mean = 12.1

b Standard deviation = 3.19, IQR = 12.8 – 10.3 = 2.5.

c $10.3 - 1.5 \times 2.5 = 6.55$. No outliers at the lower end. $12.8 + 1.5 \times 2.5 = 16.55$. So, 24.8 is an outlier.

d Mean = 11.5 and standard deviation = 1.62. The mean is closer to the median and the standard deviation is smaller, which implies that the data are less spread out.

2 **a** 6

b Mean = 5.83 and standard deviation = 1.01

c Median = 6 and IQR = 6 – 5 = 1

3 **a** $x = 5$

b $1 < t \leq 2$

c 1.5

d Mean = 2.3 and standard deviation = 1.30

e Because you do not know the exact values.

4 **a** A: New mean = 74 and standard deviation = 8

B: New mean = 71.4 and standard deviation = 8.8

C: New mean = 68.8 and standard deviation = 9.6

b Simon: A = 90, B = 89, C = 88

Millie: A = 50, B = 45, C = 40

c They will prefer A because that gives them the highest mark out of 100.

5 $70 \times 17 - 68 \times 16 = 102$

6 Sum of 10 boys = 15.3, sum of 15 girls = 23.4. So, sum of all 25 = 38.7. Mean = 1.548#

Boys: $0.1^2 = \frac{\sum x^2}{10} - 1.53^2$. So, sum of square of boys' heights = 23.509

Girls: $0.08^2 = \frac{\sum x^2}{15} - 1.56^2$. So, sum of square of girls' heights = 36.6

Total sum of squares = 60.109. variance = $\frac{60.109}{25} - 1.548^2 = 0.008056$

Therefore, standard deviation = 0.0898.

2.3 Ways in which you can present data

1 Mr Hall trains the local boys' football team and Mrs Hall trains the local girls' football team. The number of goals scored this season were:

Boys	6	2	1	1	3	0	2	0	4	1	2	2	0	0	1	5	2	0	4	3	6	1	0	0	3	5
Girls	0	2	1	5	4	1	3	2	0	7	1	6	4	0	1	2	3	5	0	7	2	1	4	3	0	1

a Complete the frequency table.

Boys	Number of goals	Girls
	0	
	1	
	2	
	3	
	4	
	5	
	6	
	7	

b Draw a frequency histogram for the boys and the girls on the same diagram and discuss any similarities or differences.

c Find the mean and standard deviation for the boys and for the girls and compare the results.

d Find the 5-number summary for the boys and for the girls and draw box-and-whisker plots.

e Compare the two box-and-whisker plots.

2 Hyeju asks 16 of her friends to count the number of steps on the staircases in their homes.

0	25	16	34	32	24	27	31
36	30	52	32	33	18	36	30

a Find the mean and standard deviation.

b Find the five-figure summary and work out if there are any outliers.

c Draw the box-and-whisker plot.

3 The length of time, t minutes, that 120 people waited in line to be served in a supermarket is shown in the table below.

Time, t minutes	Frequency
$0 < t \leq 1$	15
$1 < t \leq 2$	26
$2 < t \leq 3$	38
$3 < t \leq 4$	13
$4 < t \leq 5$	12
$5 < t \leq 6$	8
$6 < t \leq 7$	5
$7 < t \leq 8$	3

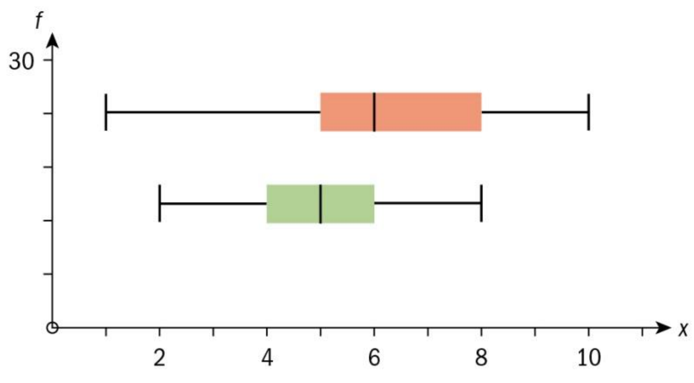
a Complete a cumulative frequency table for this data.

b Draw the cumulative frequency curve.

c Use your curve to find an estimate of the median and interquartile range.

d Use your curve to fine the 15$^{\text{th}}$ percentile.

4 At a basketball training session 40 boys and 40 girls are asked to throw the ball into the basket. They each have 10 throws. The box-and-whisker plots show the number of baskets scored.



a Find the percentage of boys who scored more than 8 baskets.

b Find the number of boys who scored 6 or fewer baskets.

c Find the number of girls who scored between 4 and 6 baskets.

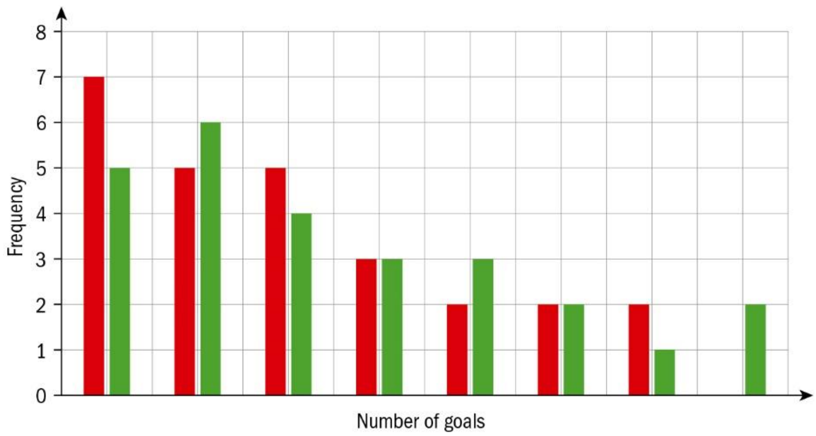
d Determine if there are any outliers.

Answers

1 a

Boys	Number of goals	Girls
7	0	5
5	1	6
5	2	4
3	3	3
2	4	3
2	5	2
2	6	1
0	7	2

b



The boys have more 0, 2 and 6 goals and the girls have more 1, 4 and 7 goals. They both have the same number for 3 and 5 goals.

c Boys' mean = 2.08 and standard deviation = 1.90

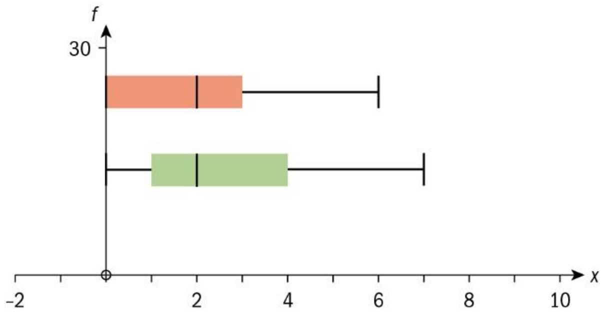
Girls' mean = 2.50 and standard deviation = 2.13

The girls have a higher mean number of goals but their data is more spread out than that of the boys.

d Boys: 0, 0, 2, 3, 6

Girls: 0, 1, 2, 4, 7

e



The girls have a larger spread. The IQR is the same for both, as is the median.

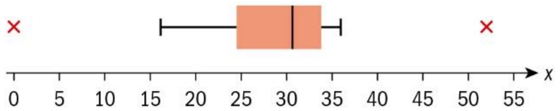
2 a Mean = 28.5 and standard deviation = 10.8

b 0, 24.5, 30.5, 33.5, 52

IQR = 33.5 – 24.5 = 9, $24.5 - 1.5 \times 9 = 11$. So, 0 is an outlier.

$33.5 + 1.5 \times 9 = 47$. So, 52 is an outlier.

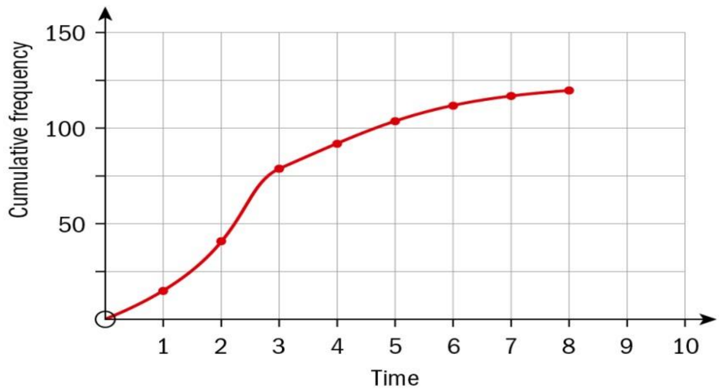
c



3 a

Time	Cumulative frequency
$t \le 1$	15
$t \le 2$	41
$t \le 3$	79
$t \le 4$	92
$t \le 5$	104
$t \le 6$	112
$t \le 7$	117
$t \le 8$	120

b



c Median approximately 2.5 and IQR approximately 3.85 – 1.58 = 2.27

d 15th percentile is at 18. So, answer is approximately 1.1

4 **a** 25%

b 25%

c 20

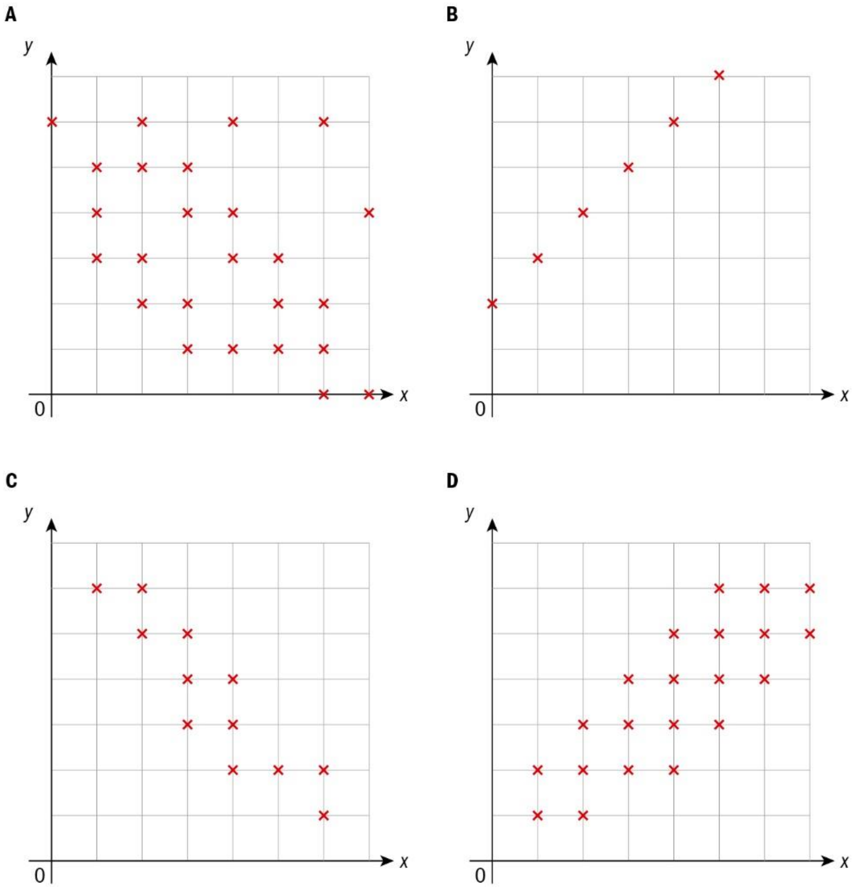
d 20

e IQR boys = 8 – 5 = 3. $5 - 1.5 \times 3 = 0.5$ and $8 + 1.5 \times 3 = 12.5$. So, no outliers.

IQR girls = 6 – 4 = 2. $4 - 1.5 \times 2 = 1$ and $6 + 1.5 \times 2 = 9$. So, also no outliers.

2.4 Bivariate data

1 Match the following diagrams to the correlation coefficients.



Correlation coefficients: 1 -0.8 0.4 -0.1

2 The following table shows the number of hours, *h*, that 14 anglers spent fishing in one week and the number of fish, *n*, that they caught.

Hours, *h*	6	12	10	18	4	6	21	15	3	10	14	8	9	18
Fish, *n*	4	9	8	13	2	3	16	14	2	6	10	6	5	15

a Plot these points on a scatter graph.

b Find Pearson's correlation coefficient, *r*, and comment on the result.

c Find the mean number of hours spent fishing and the mean number of fish caught.

d Plot the mean point on your scatter graph and draw the line of best fit by eye.

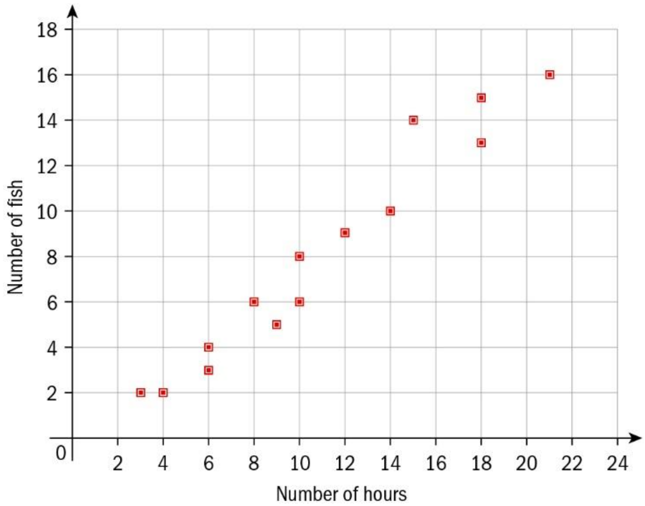
e Use your line to find the expected number of fish caught by an angler who spent 20 hours fishing in one week.

f Can you use your line to find the expected number of fish caught by an angler who spends 25 hours fishing in one week? Explain your answer.

Answers

1 A = -0.1, B = 1, C = -0.8 and D = 0.4

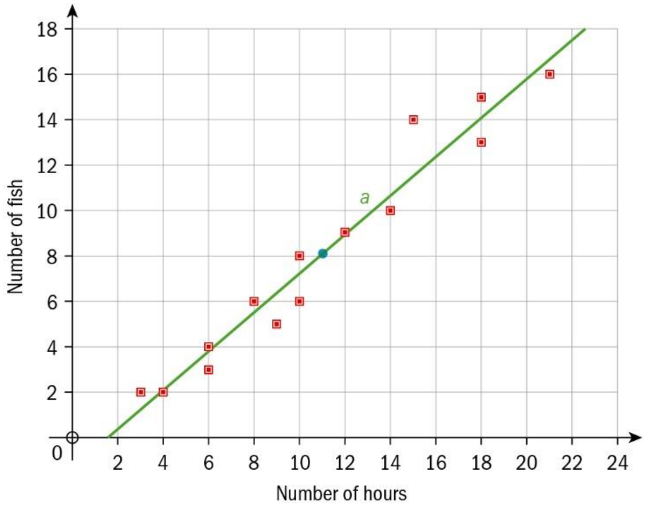
2 a



b 0.976 which is a strong and positive correlation

c Mean hours = 11 and mean number of fish = 8.07

d



e About 16 fish

f No, that would be extrapolation.

3.1 Coordinate geometry in 2 and 3 dimensions

1 A treasure has been hidden somewhere in a 3D world. Find its 3D position. To situate the starting point, find the right-angled triangle ABC among the following options and state its coordinates:

a A(3, 0) B(4, 0) C(1, 2)

b A(-6, -2) B(-4, 3) C(-2, 8)

c A(-2, 1) B(1,- 2) C(3, 0)

d $A(-2, -2\sqrt{3})$ $B(4, 0)$ $C(-2,\ 2\sqrt{3})$

e $A(1, -\sqrt{2})$ B(1, $\sqrt{2}$) C($1+\sqrt{5}$, 0)

f $A(\sqrt{7}, 4)$ $B(2\sqrt{7}, 2)$ $C(\sqrt{7}, 2)$

i Find AB, BC, AC for each

ii State whether the triangles are equilateral, isosceles or scalene

iii State whether any of ∠ABC, ∠BAC and ∠CAB is a right-angle

iv State the coordinates of the vertex of 90° angle

2 Then calculate:

a the coordinates of D such that D is on AD = $\frac{1}{3}$AC

b the coordinates of E the midpoint of DB

c the coordinates of F (a, -3.5), such that E and F are 5 units apart

d the coordinates of G such that ABCG forms a rectangle

e the coordinates of H such that HGC is isosceles and H is $\sqrt{29}$ units away from A

f the coordinates of I such as F is the midpoint of HI

3 Now let's rise above the floor and find

a the distance to I, whose z coordinates are 0 and (-2, 4, 12)

b the coordinates of J the midpoint between I and (-4,8,12)

c the coordinates of point K such that K is in the form (5, -a, 3a) and JK = 13

d the coordinates of point L such that KL is the diameter of a sphere of radius 13 and centre J

e the coordinates of point M such that MZLK is a parallelogram and Z= (3, 0, -6)

f the coordinates of point N such that NYKM is a square and Y= (7, -2, 18)

g the coordinates of O, the position of the treasure, which is the midpoint of NL

h the coordinates of the cube KMNYSPQR such that the face OPQR and thus the vertices O,P,Q and R are further away from J than the face KMNY.

Answers

1 **i – iv**

Triangle	AB	BC	AC		
a	1	$\sqrt{6}$	$2\sqrt{2}$		
b	$\sqrt{29}$	$2\sqrt{29}$	$\sqrt{29}$	Straight line	
c	$\sqrt{18}$	$\sqrt{26}$	$\sqrt{8}$	Right-angled	∠BAC
d	$\sqrt{48} = 4\sqrt{3}$	$\sqrt{48}$	$\sqrt{48}$	Equilateral	
e	$2\sqrt{2}$	$\sqrt{7}$	$2\sqrt{2}$	Isoceles	
f	$\sqrt{11}$	$\sqrt{7}$	2		

The starting point is A(-2,1)

2 **a** D = (0, -1)

b E= (1.5, -0.5)

c $5^2 - (-3.5)^2 = a^2$ so F = (5.5, -3.5)

d G = (-2 + 2, 1 + 2) = (0, 3)

e H = (3, 3, 0)

f I = (8, -10)

3 **a** $2\sqrt{110}$

b J = (2, -2,6)

c K = (5, -6, 18)

d L = (-1, 2, -6)

e M = (9, -8, 18)

f N(11, -4, 18)

g O(5, -1, 6)

h S= (7, -2, $\sqrt{505}$), P = (5, -6, $\sqrt{505}$), Q = (9, -8, $\sqrt{505}$) and R = (11, -4, $\sqrt{505}$)

3.2 The equation of a straight line in 2 dimensions

The movements of a robot football match have been mapped on a set of axes where (0,0) is the centre of the pitch and the following equations represent the movement of the ball or the robot.

1 Find the equation of the line:

a through (6, 0) and (0, 6)

b with gradient 1 and the same *x*-intercept as the line drawn in part **a**

c with gradient $\frac{1}{7}$ through (3, 3)

d through (-2, -5) and (11, 6)

e with gradient $\frac{-3}{7}$ through (15, -1).

2 Identify among the following equations:

a the parallel movements

b the perpendicular movements.

i $y = 2x + 5$

ii $y = 5x + 2$

iii $y - 5x = 2$

iv $y \;\; -7 = 2x$

v $y \;\; -2x = 6$

vi $y + 0.2x = 6$

vii $y = 2x - 5$

viii $y = 2$

ix $y = 2x$

x $y = 5x$

xi $y = 0.5x$

xii $y = 0.5x$

xiii $2x = y$

xiv $x = \frac{1}{5}y$

3 **a** Show that a robot going in a straight line through the points (-4, 3) and (11, 6) will meet a robot going through the point (1, 2) and (2, 5).

b Find the equation representing their movements.

c Calculate the coordinates they meet.

4 The ball is moving in the direction $y = 4x + 3$. A robot is standing at (-1, 3)

a Find the equation for the shortest pass to the line representing the movement of the ball.

b Find the coordinate where the robot will catch the ball.

5 Two robots moving in perpendicularly met at (4,-4) if one of the robots came from (3,6),

a Find the equation of the second robot.

b Find the two possible original positions of the second robot if he travels as far as the other robot to reach (4, -4)

6 A robot comes in the direction $\boldsymbol{y + \frac{1}{8}x = -3}$. The goalkeeper who is standing on (-10,0) wants to send him the ball making the shortest distance.

a Find the equation of the ball making shortest pass between the goal keeper and the other robot.

b Find the coordinate where the robot will catch the ball.

7 The goal keeper only moves on the line $\boldsymbol{x = 15}$ two strikers are located at S_1 (13,3) and S_2 (12,-2). Find the best position to stand so that the combined distance from its distance from both strikers is least.

8 The referee moves only moves along the line $y = 4$. The two furthest players are located at P_1 (-8, 2) and P_2 (10, -2). Find the best position to stand so that the combined distance from the referee's position to both players is least.

9 The goal is located at G (-13, 0), and a striker is placed at S (-9, 2). If the defence robot can only move along the line $y = \frac{1}{2}x + 5$, find the position where the combined distance from the goal or the striker is the least.

10 The equations of the movements of balls are as follow. Find the resulting shape.

$y = 2$, $y = 6.5$, $x = -1.2$, $x = 3.1$

$y = 2$, $y = 6.5$, $y = 2x + 3$, $y = 2x - 3$

$x = 2$, $x = -6.5$, $y = 2x + 3$, $y = 3x$

Answers

1 **a** $y = -x + 6$

b $y = -x + 6$

c $y = \frac{1}{7}x + 2\frac{4}{7}$ or $y = \frac{1}{7}x + \frac{18}{7}$

d $y = \frac{5}{7}x - \frac{18}{7}$

e $y = \frac{-3}{7}x - \frac{38}{7}$

2 **a** a, d, e, g, i and m or b, c, j and n

b a and l, a and f

3 **a** As their gradients are different, i.e. $\frac{3}{15} = \frac{1}{5} \neq 3$, they will meet.

b $y = \frac{1}{5}x + \frac{19}{5}$ and $\boldsymbol{y} = 3x - 1$

c $(\frac{12}{7}, \frac{29}{7})$

4 **a** $y = -\frac{\mathbf{1}}{4}x + 2.75$

b $(-\frac{2}{30}, \frac{82}{30})$

5 **a** $y = \frac{1}{10}x - 4.4$

b (14, -3) or (-6, -5)

6 **a** $y = 8x - 10$

b $(\frac{56}{65}, \frac{-202}{65})$

7 The shortest distance from S_1 to the line is at (15, -2), so the image of S_1 reflected in the line is S_1'(18, -2). The equation of S_1' S_2 is $\boldsymbol{y} = -x + 23$. S_1' S_2 cuts the x = 15 at (15, 1). Therefore, (15,1) is the best position.

8 The shortest point from P_2 to the line is at (10,4), so the image of P_2 reflected in the line is P_2'(10, 8). The equation of P_2' P_1 $\boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{x} + \frac{\mathbf{14}}{\mathbf{3}}$ so P_2' P_1 cuts the y=4 at (-2, 4). Therefore, (-2, 4) is the best position.

9 The shortest point from G to the line is at (-12,-1), so the image of G reflected in the line is G'(-11, -2). The equation of G' S $\boldsymbol{y} = \mathbf{2}\boldsymbol{x} + 20$. G'S cuts $\boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x} + \mathbf{5}$ at (-10, 0) so (-10,0) is the best position.

10 **a** Two sets of parallel lines (horizontal and vertical) with same length and a perpendicular angle make a square.

b Two sets of parallel lines (horizontal and diagonal e.g. y=2x+ …) make a parallelogram.

c One set of parallel lines (vertical) and the other sides not parallel make a trapezium.

3.3 Voronoi diagrams

1. To preserve the wellbeing of the animals, the local safari park has decided that cars won't be allowed to come too close to the watering holes. For this reason, they want to draw compulsory tracks that visitors will have to stay on.

The natural park can be considered as a pentagon. A coordinate grid is placed on a map of the park such that the borders of the park can be modelled by $y_1 = 0.2x + 12, y_2 = x - 3x + 44, y_3 = 0.8x - 13, y_4 = -9, x = -10,$ and the watering holes are situated at $A(\ 6{,}8), B(\ -4{,}6), C(4{,}2), D(-6{,}2), E(-2{,}6),\ F(\ 12{,}0), G(11.6)$ and all the measurements are in km^2

a Draw the Voronoi diagram representing the map of the park borders, the watering holes and eventually the new track, the resulting new entrance gates and the guard house after answering the following:

b Find:

i the midpoint of [AB]

ii the gradient of [AB].

iii Hence find the equation of the perpendicular bisector of [AB].

iv Find the coordinates of the new gate, which is at the intersection with y_1

c Find:

i the midpoint of [AC]

ii the gradient of [AC]

iii the equation of the perpendicular bisector of [AC].

d Calculate the coordinates of J, the location of the first crossroad i.e. the point of intersection of the two bisectors.

e Using a similar method, find the equation of the perpendicular bisector of:

i [BC]

ii [BD]

iii [CD].

iv Hence find K, the second crossroad.

f Show that the perpendicular bisectors of [BC] and [BD] are perpendicular.

g Using a similar method, find the equation of the perpendicular bisector of:

i [DE]

ii [CE]

iii [CF]

iv [EF]

v [CG]

vi [GA]

vii [FG].

h Hence find:

i L, M, N and P, the other crossroads

ii the coordinates of the 5 remaining gates, the intersection of the perpendicular bisectors leading to a border and the closest border

iii the total length of the road to be created.

i A ranger house is to be built in the park as far away as possible from any watering hole. State where it should be built. A close observation program will be run in the area around D. Calculate:

i. the area of D

ii the area of the park.

iii the percentage of the area of the park it represents.

2 In the area of D, 4 x $360°$ cameras have been placed at AD (-9, -2), BD (-5, 2), CD (-3, -2), DD (-9, 3), respectively. A fifth needs to be hung to stand as far away as possible from the others. State:

a its position, using a Voronoi diagram

b the probability of catching a photo of a lion at D if the probability in the area AD is 15%, BD is 75%, CD is 20% and DD is 50%

c the area covered by AD and CD

d the mean probability of seeing a lion in area D if area BD = 20.18 km^2 and area BD=21.46 km^2.

Answers

1 **b** **i** (1,7)

ii $\frac{1}{5}$

iii $y = -5x + 12$

iv (0,12)

c **i** (5,5),

ii 3

iii $y = -\frac{1}{3}x + 6\frac{2}{3}$

d $-5x + 12 = -\frac{1}{3}x + 6\frac{2}{3}$,$-15x + 36 = -1x + 20$ so $x = -\frac{16}{14} = 1\frac{1}{7}$

Substituting into $-5x + 12$, $y = 6\frac{2}{7}$ so J$(1\frac{1}{7}, 6\frac{2}{7})$.

e **i** $y = 2x + 4$

ii $y = -0.5x + 1.5$

iii $x = -1$

iv K (-1,2)

f The gradient of the perpendicular bisectors of [BC] =2. As $-\frac{1}{2}$ = -0.5 = the gradient of the perpendicular bisectors of [BD] then they are perpendicular.

g **i** ⊥[DE] $y = x$

ii $\perp$ [CE] $y = -0.25x - 1.25$

iii $\perp$ [CF] $y = 4x - 31$

iv $\perp$ [EF] $y = -1\frac{2}{3}x + 8\frac{2}{3}$

v $\perp$ [CG] $y = -1\frac{3}{4}x + 17\frac{1}{8}$

vi $\perp$ [GA] $y = 2.5x - 14\frac{1}{4}$

vii $\perp$ [FG] $y = \frac{1}{6}x + 1\frac{25}{300}$

h **i** L (-1,-1), M(7,-3), N$\left(8\frac{17}{46}, 2\frac{11}{23}\right)$, P $\left(7\frac{13}{34}, 4\frac{7}{34}\right)$

ii (10.59,12.23), (13.55, 3.34), (8.78, -5.97),(-9,-9), (-10,6.5)

iii $B_1K = 10.6, C_1J = 5.83, D_1D = 8.64, VN = 5.25, WM = 3.47, A_1L = 11.31, LK = 3, KJ = 4.79, JP = 6.58, PN = 1.99, NM = 5.65, ML = 8.25$

So the total length = 74.82 km

i It can be built at M or L as $CM = CL = 5.83 > CJ, CK, CP, CN$

j **i** 87.25

ii 457.5

iii 19%

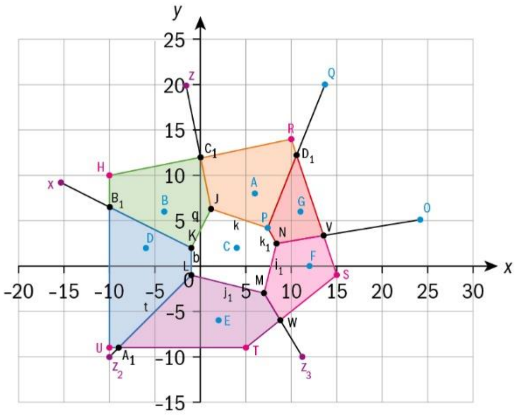
2 **a** E = (-7.5, 0.5), G = (-6, -1) as BDGD = 3.16 and BDED =2.91. GD is the best place to position the router

b 75%

c AD = 41.115 km^2 and CD = 18.75 km^2

d $\frac{(41.115 \times .15 + 20.18 \times .75 + 18.75 \times .2 + 21.46 \times .5)}{41.115 + 20.18 + 18.75 + 21.46} = 35.25\%$

https://www.geogebra.org/graphing/nz5sxybt



https://www.geogebra.org/graphing/ub2zyevz

3.4 Displacement vectors

1 A laser-cutter has been programmed to engrave pieces of wood following the 3 set vectors **a, b** and **c** to create a geometrical bird.

a The bird's beak is an equilateral triangle *ABC* with point *CM* half way along *CB* to represent the beak. If $\overrightarrow{AC}$ has been programmed as **a** and $\overrightarrow{BA}$ as **b**, find**:**

i $\overrightarrow{CB}$

ii $\overrightarrow{CM}$

iii $\overrightarrow{AM}$ in terms of **a** and **b**

iv x if **a** = $\begin{pmatrix} x \\ \frac{1}{2} \end{pmatrix}$**,** **b** = $\begin{pmatrix} -x \\ \frac{1}{2} \end{pmatrix}$ and $x > 0$.

b Its head is the regular hexagon BCDEFG with centre *H.* Find:

i $\overrightarrow{BH}$, $\overrightarrow{BG}$, $\overrightarrow{GD}$, $\overrightarrow{EB}$ and $\overrightarrow{EF}$ in terms of **a** and **b**

ii $\overrightarrow{BH}$, $\overrightarrow{BG}$, $\overrightarrow{GD}$, $\overrightarrow{EB}$ and $\overrightarrow{EF}$ in the form $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$

iii the magnitude of $\overrightarrow{BH}$, $\overrightarrow{BG}$, $\overrightarrow{GD}$, $\overrightarrow{EB}$ and $\overrightarrow{EF}$

iv the position vector of C, D, E, F, G in terms of $\sqrt{3}$ if $\overrightarrow{OB} = \begin{pmatrix} \mathbf{1} \\ \mathbf{6.5} \end{pmatrix}$

c Its body is a trapezium *GHIJ* with *IJ* parallel to *G*, $\overrightarrow{LK} = \frac{5}{3}\overrightarrow{IJ}$, $\overrightarrow{OJ} = \begin{pmatrix} \mathbf{2.5} \\ \mathbf{5} \end{pmatrix}$ and $\overrightarrow{OK} = \begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$**.** Write $\overrightarrow{JK}$:

i in component form

ii unit vector form

iii in the form $\overrightarrow{JK} = k\mathbf{c}$ where $k \in \mathbb{R}^+$and **c** is a unit vector.

d $|\overrightarrow{JL}| = \sqrt{5}|\boldsymbol{c}|$ and $\overrightarrow{KL} = m$ (-**a** + **b**) $m > 1$. Find:

i m

ii $\overrightarrow{OL}$

iii $\overrightarrow{KL}$

iv $\overrightarrow{OI}$.

e Point R cuts the line LJ in the ratio 4:1. Find**:**

i $\overrightarrow{LR}$ in component form

ii $\overrightarrow{LR}$ in terms of **a, b** and **c**

iii $\overrightarrow{OR}$.

f P is the midpoint of LP.

i Find $\overrightarrow{LP}$ and $\overrightarrow{KP}$.

ii Describe the geometrical relationship between $\overrightarrow{LP}$ and $\overrightarrow{KP}$.

ii Write $\overrightarrow{LP}$ and $\overrightarrow{KP}$ in terms of **a, b** and **c**.

g The eyes are a triangle MNQ with M=(1.5, 6.5) , $\overrightarrow{MN} = -\frac{1}{10}, \overrightarrow{KP}$, $\overrightarrow{QM} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{q} \end{pmatrix}$ **,** $\overrightarrow{QN} = \begin{pmatrix} \boldsymbol{q} \\ \mathbf{0.1} \end{pmatrix}$ and $|\overrightarrow{QM}| = \frac{2}{\sqrt{5}}|\overrightarrow{QN}|$.

i Find the coordinates of N and Q.

ii Describe the geometrical property of ΔMNQ.

h The leg will be made of three engraved segments [ST], [TU] and [TV] with T= (2 ,2.5).

i Find S, the point on [LK] closest to T.

ii Decide whether, if $\overrightarrow{TU} \perp \overrightarrow{KP}$ and $|\overrightarrow{TU}| = |\overrightarrow{QN}|$, we can conclude that $\overrightarrow{TU} = \overrightarrow{MN}$.

iii The position vectors for U and V have the same *y*-coordinate, and state the coordinate of U.

iv Prove that K, T and V are collinear.

2 In a 3D printer, the extruder can move in the three directions **i**, **j**, **k** to create the decoration for a quadrilateral standing frame AEFG.

a Write

i **a=** 12**i**+ m**j** +n**k** in component form

ii **b**= 132**i**+ 165**j** +176**k** as a scalar of the unit vector.

b Find m and n if **a** and **b** are parallel.

c Let B, C and D, the positions of decorations, be equally spread on AE with $\overrightarrow{AE}$ = **a** and $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, calculate their coordinates.

d AEFG is a parallelogram with $\overrightarrow{EF} = \begin{pmatrix} -1 \\ 4 \\ -8 \end{pmatrix}$. Find the coordinates of E, F and G.

e Find the length of EF.

Answers

1 a i $\overrightarrow{CB} = \mathbf{a} + \mathbf{b}$

ii $\overrightarrow{CM} = \frac{\mathbf{a}+\mathbf{b}}{2}$

iii $\overrightarrow{AM} = \frac{\mathbf{b}-\mathbf{a}}{2}$

iv $\frac{\sqrt{3}}{2}$

b i $\overrightarrow{BH} = \mathbf{a}$,. $\overrightarrow{BG} = -\mathbf{b}$, $\overrightarrow{GD} = 2(\mathbf{a}+\mathbf{b})$, $\overrightarrow{EE} = -2\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{a}-\mathbf{b}$

ii $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

iii 1, 1, 2, 2, 1

iv $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 6.5 \end{pmatrix}$, $\overrightarrow{OD} = \begin{pmatrix} 1+\frac{\sqrt{3}}{2} \\ 7 \end{pmatrix}$, $\overrightarrow{OE} = \begin{pmatrix} 1+\sqrt{3} \\ 6.5 \end{pmatrix}$, $\overrightarrow{OF} = \begin{pmatrix} 1+\sqrt{3} \\ 5.5 \end{pmatrix}$, $\overrightarrow{OG} = \begin{pmatrix} 1+\frac{\sqrt{3}}{2} \\ 5 \end{pmatrix}$,

c i component form = $\begin{pmatrix} 1.5 \\ -2 \end{pmatrix}$

ii 1.5**i**-2**j**

iii k = 2.5 **c** = $\begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}$

d i $|\overrightarrow{JL}| = \begin{pmatrix} x-2.5 \\ y-5 \end{pmatrix} = \sqrt{5}$,

$\overrightarrow{KL} = m$ (-**a**+**b**)$= m\begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-3 \end{pmatrix}$

so $y = 3$ and

$\left|\begin{matrix} x-2.5 \\ -2 \end{matrix}\right| = \sqrt{5}$ so $x = 1.5$ **or 3.5**

if $(3.5-4) = -\sqrt{3}\,m$ **is** $m < 1$

$m(1.5-4) = -\sqrt{3}\,m$ $m = \frac{5}{2}\sqrt{3}$

ii $\overrightarrow{OL} = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$,

iii $\overrightarrow{KL} = \begin{pmatrix} -2.5 \\ 0 \end{pmatrix}$,

iv $\overrightarrow{IJ} = \frac{3}{5}\begin{pmatrix} 2.5 \\ 0 \end{pmatrix}$ **so** $\overrightarrow{OI} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

e i 0.8i +1.6**j**

ii $\frac{4}{5}\left(\frac{5}{2}\sqrt{3}(\mathbf{a}-\mathbf{b}) - \frac{5}{2}\mathbf{c}\right) = 2\sqrt{3}(\mathbf{b}-\mathbf{a}) - 2\mathbf{c}$

iii $\begin{pmatrix} 2.3 \\ 4.6 \end{pmatrix}$

f i $\overrightarrow{LP} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$, $\overrightarrow{KP} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

ii $\overrightarrow{LP} = \begin{pmatrix} \mathbf{0.5} \\ \mathbf{1} \end{pmatrix}, \overrightarrow{KP} = 2\begin{pmatrix} \mathbf{-1} \\ \mathbf{0.5} \end{pmatrix}$ so $\overrightarrow{KP} \perp \overrightarrow{LP}$ and the magnitude of $\overrightarrow{KP}$ is twice $\overrightarrow{LP}$

iii $\overrightarrow{LP} = \frac{\mathbf{5}}{\mathbf{4}}\sqrt{\mathbf{3}}(\mathbf{a}-\mathbf{b}) - \frac{5}{4}\mathbf{c},\ \overrightarrow{KP} = -\frac{\mathbf{5}}{\mathbf{4}}\sqrt{\mathbf{3}}(\mathbf{a}-\mathbf{b}) - \frac{5}{4}\mathbf{c}$

g **i** N (1.7, 6.4), Q (1.5, 6.3)

ii $\overrightarrow{MN} = \begin{pmatrix} \mathbf{0.2} \\ \mathbf{-0.1} \end{pmatrix}, \overrightarrow{QN} = \begin{pmatrix} \mathbf{0.2} \\ \mathbf{0.1} \end{pmatrix}$ so $|\overrightarrow{MN}| = |\overrightarrow{QN}|$ so ΔMNQ is isosceles

h **i** If S is the point on [LK] closest to T, $\angle KST = 90°$ so S(2,3)

ii If $\overrightarrow{TU} \perp \overrightarrow{KP}$, $\overrightarrow{TU} // \overrightarrow{MN}$

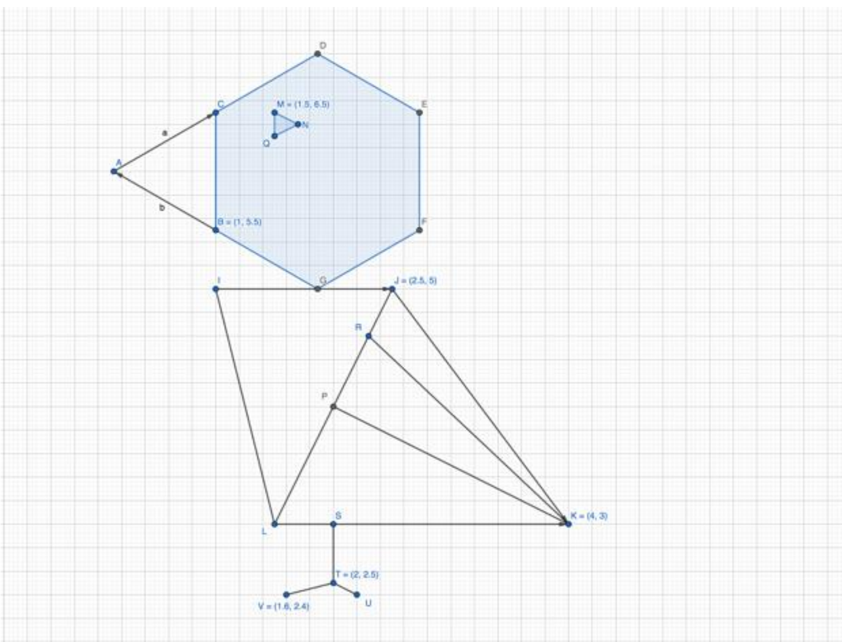
$|\overrightarrow{TU}| = |\overrightarrow{QN}| = |\overrightarrow{MN}|$

The two vectors are parallel with the same magnitude but we are not sure about the direction.

So $\overrightarrow{TU} = \overrightarrow{MN}\ or\ \overrightarrow{TU} = \overrightarrow{-MN}$

iii (2.2, 2.4)

iv $\overrightarrow{VT} = \begin{pmatrix} \mathbf{0.4} \\ \mathbf{0.1} \end{pmatrix}, \overrightarrow{TK} = \begin{pmatrix} \mathbf{2} \\ \mathbf{0.5} \end{pmatrix} = 0.5\overrightarrow{VT}$



Copy of geogebra: https://www.geogebra.org/graphing/mmfakktd

2 **a** **i** $\begin{pmatrix} 12 \\ m \\ n \end{pmatrix}$

ii $\left|\begin{pmatrix} 132 \\ 165 \\ 176 \end{pmatrix}\right|$=275 so **b** = 275. $\begin{pmatrix} \frac{12}{25} \\ \frac{3}{5} \\ \frac{16}{25} \end{pmatrix}$

b **a=** 12**i**+ 15**j** +16**k**

c $\overrightarrow{AB}=\begin{pmatrix}3\\3.75\\4\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}6\\7.5\\8\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}9\\11.25\\12\end{pmatrix}$

B = (4, 5.75, 7), C= (7, 9.5, 11), D = (10, 13.25, 15)

d E = (13, 17, 19), $\overrightarrow{OF}=\overrightarrow{OE}+\overrightarrow{EF}=\begin{pmatrix}13-1\\17+4\\19-8\end{pmatrix}=\begin{pmatrix}12\\21\\11\end{pmatrix}$, $\overrightarrow{OG}=\overrightarrow{OA}+\overrightarrow{EF}=\begin{pmatrix}1-1\\2+4\\3-8\end{pmatrix}=\begin{pmatrix}0\\6\\-5\end{pmatrix}$

e 9

3.5 The scalar and vector products

T

In 2010 Studio Pacific Architecture and Warren and Mahoney redesigned Wellington International Airport using a mesmerising three-dimensional mosaic of wooden panels as seen below. We will analyse a model of the picture below to find out the surface area and angles which creates the beauty of the room. For this we will set the origin O as the point right below A at floor level and the distance in metres. https://www.geogebra.org/graphing/chy5gnk2

1 Let A = (0, 0, 6), B = (-2, -1, 7) and C= (-1, -3, 10)

a To find $\angle ABC$, **discuss** whether using $\overrightarrow{BA}$ and $\overrightarrow{BC}$ will give you the same as using $\overrightarrow{AB}$ and $\overrightarrow{BC}$

b Calculate:

i $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in vector form

ii $\overrightarrow{AB} \bullet \overrightarrow{BC}$

iii $|\overrightarrow{BA}|$ and $|\overrightarrow{BC}|$

iv $\angle ABC$

v $\overrightarrow{AB} \times \overrightarrow{BC}$

vi the surface area of the panel ABC

vii $\angle BCA$

viii d in $\overrightarrow{CD} = \begin{pmatrix} 6 \\ d \\ -1 \end{pmatrix}$ if area of ACD = $2.5\sqrt{42}$

ix $\angle ACD$

x $\angle BCD$ and explain why $\angle ACD + \angle BCA \neq \angle BCD$

xi a if $\overrightarrow{AE} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} -\frac{a}{2} \\ -a^3 \\ a^2 \end{pmatrix}$ are perpendicular and $|\overrightarrow{EF}| = 3$

c Find the formula:

i for the lateral surface area of a hexagonal pyramid where the apex is the origin and **a, b, c, d, e** and **f** are each the position vertex of the vertices B, H, I, J, K and L

ii the total surface area of the hexagonal pyramid.

2 Outside the room the decoration is made of prisms with all the edges represented by vectors **a, b, c**.

a Calculate all the possible vectors perpendicular to

i **a** = $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and **b** = $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

ii **a** = -**I** - 2**j** and **b** = 3**i** + 5**j** - 4**k**

iii **a**=$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and **b**=$\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ and with magnitude $\sqrt{14}$ and show that they are all perpendicular to each other.

Answers:

1 a No, you get the exterior angle i.e. 180 - $\angle ABC$

b i $\overrightarrow{BA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

ii -3

iii $|\overrightarrow{BA}| = \sqrt{6}$ and $|\overrightarrow{BC}| = \sqrt{14}$

iv $\angle ABC = 109°$

v **i**-7**j**-5**k**

vi $5\sqrt{3}$

vii $\overrightarrow{CA} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}, \overrightarrow{CA} \bullet \overrightarrow{CB} = 17, |\overrightarrow{CA}| = \sqrt{26}$ so $\angle BCA = 27°$

viii $\overrightarrow{CA} \times \overrightarrow{CD} = (-4d+3)\boldsymbol{i} + 23\boldsymbol{j} + (18-d)\boldsymbol{k}$

$(-4d+3)^2 + 23^2 + (18-d)^2 = 1050$ so d = -2

ix $\overrightarrow{CA} \bullet \overrightarrow{CD} = 4,\ |\overrightarrow{CD}|,$ so $\angle ACD = \sqrt{41}\ 83°$

x $\overrightarrow{CB} \bullet \overrightarrow{CD} = -7,$ so $\angle BCD = 107°$

xi a = -2

c i $\frac{1}{2}(|\mathbf{a} \times \mathbf{b}| + |\mathbf{b} \times \mathbf{c}| + |\mathbf{c} \times \mathbf{d}| + |\mathbf{d} \times \mathbf{e}| + |\mathbf{e} \times \mathbf{f}| + |\mathbf{f} \times \mathbf{a}|)$

ii $\frac{1}{2}(|\mathbf{a} \times \mathbf{b}| + |\mathbf{b} \times \mathbf{c}| + |\mathbf{c} \times \mathbf{d}| + |\mathbf{d} \times \mathbf{e}| + |\mathbf{e} \times \mathbf{f}| + |\mathbf{f} \times \mathbf{a}| + |(\mathbf{b} - \mathbf{a}) \times (\mathbf{f} - \mathbf{a})| + |(\mathbf{a} - \mathbf{e}) \times (\mathbf{b} - \mathbf{e})| + |(\mathbf{e} - \mathbf{d}) \times (\mathbf{b} - \mathbf{d})| + |(\mathbf{d} - \mathbf{c}) \times (\mathbf{b} - \mathbf{c})|)$

2 a i $p\begin{pmatrix} -5 \\ -7 \\ -3 \end{pmatrix}$ where $p \in \mathbb{R}^*$

$p\begin{pmatrix} -8 \\ -4 \\ -11 \end{pmatrix}$ where $p \in \mathbb{R}^*$

$\pm\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0$, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} = 0$ and $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} = 0$

b i $m = 0, 5\ and\ -14$

ii $m = 1$ **a =** $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ **, b =** $\begin{pmatrix} 23 \\ 8 \\ 11 \end{pmatrix}$**, c =** $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

c if **A, B, C** are collinear $\overrightarrow{AB} = k\,\overrightarrow{AC},\ \overrightarrow{BC} = p\,\overrightarrow{AC}$

$\overrightarrow{AB} \times \overrightarrow{BC} = k\,\overrightarrow{AC} \times p\overrightarrow{AC} = kp\,\overrightarrow{AC} \times \overrightarrow{AC} = kp\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

3.6 Vector equations of lines

Tianmen mountain (China) is famous for 'the longest passenger cableway of high mountains in the world'.

1 If the cable car were going in a straight line to the top, the movement of a cabin would be modelled by $x(t) =$ -0 and $y(t) = -6.25t + 8{,}000$ and $z(t) = t - 1280$ with $t \geq 0$ in s, and x, y and z the distance in metres eastbound, northbound, and vertically from the top station of the cable car.

a Write the cabin's movement as a vector equation.

b Find the cabin's:

i position at the bottom station

ii position after 3 m

iii speed.

2 The cabin actually goes over a small hill first, reaches the middle station (0, 900, -600) after 20 minutes, and from there it starts its final leg at an average speed of 3.6 m/s.

a Find:

i the cabin's direction vector

ii the cabin's angle of depression

iii the cabin's velocity vector

iv the cabin's vector equation in term of $(t - 20)$

v the cabin's position when t = 0, hence its vector equation in term of t

vi the time the cabin takes to reach the top station.

The World Wingsuit League is held on the same mountain.

b A wingsuiter has a new suit, which enables him to maintain his speed once it is over 100 km/h. He wants to fly as closely as possible to the cable car. Select the possible equations this scenario could take and justify your choice.

i $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 10 \\ 970 \\ -650 \end{pmatrix}$

ii $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \\ -20 \end{pmatrix} t + \begin{pmatrix} 10 \\ 960 \\ -640 \end{pmatrix}$

iii $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 20 \end{pmatrix} t + \begin{pmatrix} 10 \\ 960 \\ -640 \end{pmatrix}$

iv $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -6 \end{pmatrix} t + \begin{pmatrix} 0 \\ 940 \\ 240 \end{pmatrix}$

v $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -36 \\ -24 \end{pmatrix} t + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$

vi $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -20 \\ -30 \end{pmatrix} t.$

3 The wingsuiter wants to jump westbound and maintain a speed of 85 km/h. Find:

a **i** his velocity vector if his vertical speed is 40 km/h

ii his angle of depression

iii the vector equation if a constant speed is reached after 36 s at (-30, 0, -16) in metres

iv the angle between his path and the path of the cable car.

v the time it would take him to be at a position (x, y, z) with $z \leq -600$

vi the resulting *x*- and *y*- coordinates.

b If he dives without allowing for a wind of $\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$, find:

i the speed of the wind

ii the wind's bearing

iii the windsurfer's new velocity vector

iv his new angle of depression

v his new overall speed

vi his new bearing.

c If he allows for the wind during his dive, find:

i the bearing needed to compensate for the wind

ii the resulting speed.

4 The wingsuiter also plans to jump from a helicopter and go through Tianmen Cave.

a If the parametric equation of his glide is $\lambda = \frac{x-3}{48} = \frac{5-y}{90} = \frac{-0.1-z}{5.4}$ in kms and hours, find:

i the Cartesian equation of the line as a vector equation

ii the velocity

iii the shortest distance to the photographer situated at (4.615, 2.008, -0.28)

iv the position vector after 5 min.

b At (7, -2.5, -0.55) he changes direction and passes through (8, 0.5, -0.65) 6 minutes later. Find:

i the new direction vector

ii the angle he turned

iii the new velocity vector

iv the vector equation in term of μ hours

Answers

1 a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -6.25 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 8000 \\ -1280 \end{pmatrix}$

b i (0, 8,000, -1,280)

ii $\begin{pmatrix} 0 \\ 8{,}000 - 6.25 \times 3 \times 60 \\ 180 - 1280 \end{pmatrix} = \begin{pmatrix} 0 \\ 6875 \\ -1100 \end{pmatrix}$

iii $\sqrt{6.25^2 + 1^2} = 6.33\text{ms}$

2 a i $\begin{pmatrix} 0 \\ -900 \\ 600 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

ii $33.7°$

iii $\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

iv $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} (t - 1{,}200) + \begin{pmatrix} 0 \\ 900 \\ -600 \end{pmatrix}$

v $\begin{pmatrix} 0 \\ 960 \\ -640 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 4{,}500 \\ -3{,}000 \end{pmatrix}$

vi 5 minutes and 20 sec, so 25 minutes and 20 sec

b i No. As it is the same direction vector, it would mean that it is flying up parallel to the cable car.

ii Yes, the wingsuiter would fly parallel to the cable car 10 m east of it at a speed of 36 m/s or 129.6 km/h

iii It is unlikely the wingsuiter would fly parallel to the cable car 10 m East of it. His speed would, however, be low.

iv No, the wingsuiter's path will intersect perpendicular the path of the cable car at i00.

v Yes, the wingsuiter's path will be perpendicular the path of the cable but 10 m East of it.

vi Yes, as the wingsuiter's path will intersect only at (0,0,0).

3 a i $\begin{pmatrix} -75 \\ 0 \\ -40 \end{pmatrix}$

ii $\sin^{-1}\left(\frac{40}{85}\right) = 28°$

iii $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{125}{6} \\ 0 \\ -\frac{100}{9} \end{pmatrix}(t - 36) + \begin{pmatrix} -30 \\ 0 \\ -16 \end{pmatrix} = \begin{pmatrix} -\frac{125}{6} \\ 0 \\ -\frac{100}{9} \end{pmatrix} t + \begin{pmatrix} 720 \\ 0 \\ 384 \end{pmatrix}$

iv 105.1°

v t ≥ 88.56s

vi x = -1125, y=0

b **i** 9 km/h

ii 26.6°

iii $\begin{pmatrix} -72 \\ 6 \\ -34 \end{pmatrix}$

iv 28.3°

v 79 km/h

vi 274.8°

c **i** He needs to fly aiming at $\begin{pmatrix} -78 \\ -6 \\ -46 \end{pmatrix}$, so 265.6°

ii 90.8 km/h

4 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 48 \\ -90 \\ -5.4 \end{pmatrix} \lambda + \begin{pmatrix} 3 \\ 5 \\ -0.1 \end{pmatrix}$

ii 102 km/h

iii $t = \frac{2}{70}$**i** 0.017 km = 17 m

iv (7, -2.5, -0.55)

b **i** $\begin{pmatrix} 1 \\ 3 \\ -0.1 \end{pmatrix}$

ii 133

iii $\begin{pmatrix} 0.1 \\ 0.3 \\ -0.01 \end{pmatrix}$

iv $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = . \begin{pmatrix} 8 \\ 0.5 \\ -0.65 \end{pmatrix} + \mu \begin{pmatrix} 0.1 \\ 0.3 \\ -0.01 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = . \begin{pmatrix} 7 \\ -2.5 \\ -0.55 \end{pmatrix} + \mu \begin{pmatrix} 0.1 \\ 0.3 \\ -0.01 \end{pmatrix}$

4.1 Functions

1 **a** In 2018, the DP Maths SL grade boundaries were as follows:

Percentage	$p \leq 16$	$16 < p \leq 33$	$33 < p \leq 47$	$47 < p \leq 58$	$58 < p \leq 70$	$70 < p \leq 82$	$82 < p \leq 100$
Grade	1	2	3	4	5	6	7

Draw the **relation** between

i the percentage and the grade

ii the grade and the percentage.

iii State whether you can precisely find your grade if you know your percentage.

iv State whether you can precisely find your percentage if you know your grade.

v State whether any of the graphs in **i** or **ii** represent a function.

vi State as true or false: i. a relation is always function; ii. a function is always a relation.

vii State the domain and the range for each *relation*.

viii Write the equations describing each *relation*.

b State whether the following sentences are true or false:

i $y = a$ is always a function where a is a constant

ii $x = a$ is always a function

iii The inverse of $x = a$ is $y = a$

iv A function always has an inverse

v A function always has an inverse function

2 The DP coordinator decided to investigate the relation between the students' Maths grade in 4 different years and 10 classes, and their results in DP in SL and HL. Here are the results.

$i.$ $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$, $(4,3)$, $(4,7)$, $(5,6)$, $(5,5)$, $(5,4)$, $(6,3)$, $(6,6)$, $(7,6)$, $(7,7)$, $(7,5)$, $(7,7)$, $(7,7)$

$ii.$ $(1,1)$, $(1,1)$, $(2,1)$, $(2,1)$, $(3,4)$, $(4,4)$, $(4,4)$, $(5,6)$, $(5,6)$, $(5,6)$, $(6,5)$, $(6,5)$, $(7,6)$, $(7,6)$, $(7,6)$, $(7,6)$

$iii.$ $(1,2)$, $(1,2)$, $(2,3)$, $(2,4)$, $(3,4)$, $(3,4)$, $(3,4)$, $(4,5)$, $(4,5)$, $(4,5)$, $(4,5)$, $(5,6)$, $(5,6)$, $(5,6)$, $(6,7)$, $(6,7)$

$iv.$ $(2,3)$, $(2,3)$, $(2,3)$, $(2,3)$, $(5,6)$, $(5,6)$, $(5,6)$, $(5,6)$, $(6,7)$, $(6,7)$, $(6,7)$, $(6,7)$

For each *relation,* state*:*

a the domain and the range .

b whether it is one-to-one, many-to-one, many-to-many or one-to-many

c whether it represents a function.

d Draw each relation and show how you can graphically verify **b** and **c**.

e Express the relations (if possible) as an equation in the form $R_k(x) = \cdots.$ where $k = i, ii, iii$ or iv.

f For all k find:

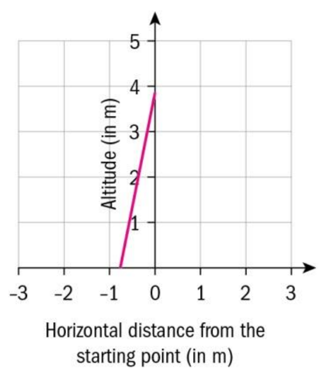
i $R_k(1)$

ii $R_k(5)$

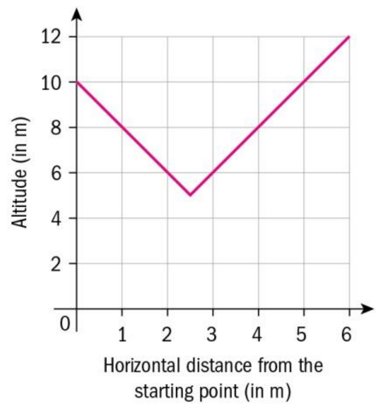
iii $R_k(x) = 7.$

3 A new extreme trail is planned to be launched in the region. It will involve extreme sports. The trail starts with climbing a ladder rope, rock-climbing to access a narrow bridge, bouncing on a trampoline, crossing a lake on a zip line, rock-climbing a negative slope, bungee jumping, being lifted up 30 metres and finally sliding down a river. For safety reasons, the management needs to ensure emergency equipment is in place. They therefore need to know the exact horizontal and vertical position of their client at any point on the trail. The trail has been modelled as follows .

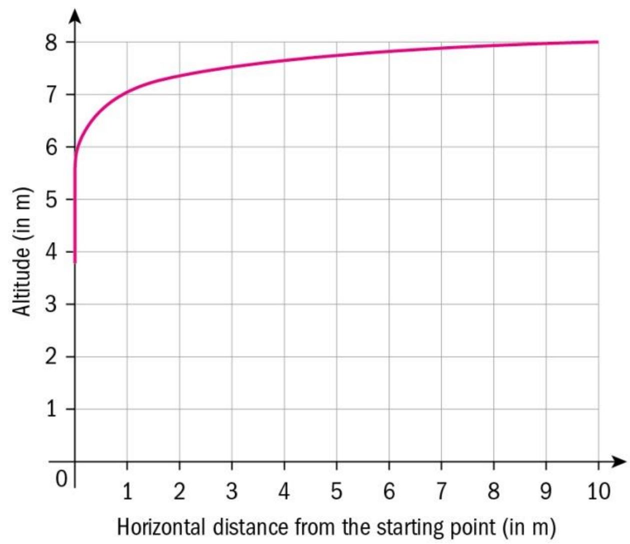
a



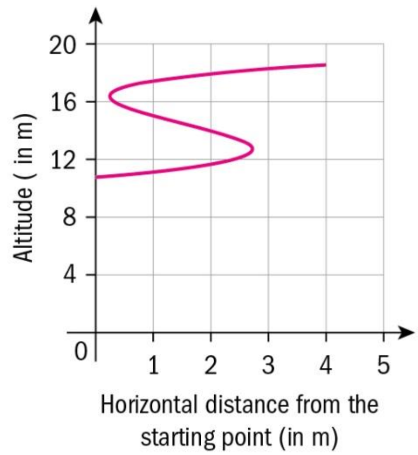
b



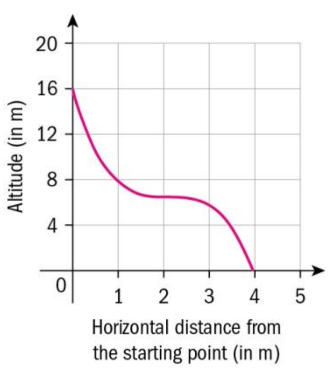
c


Altitude (in m)
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 9 10
Horizontal distance from the starting point (in m)

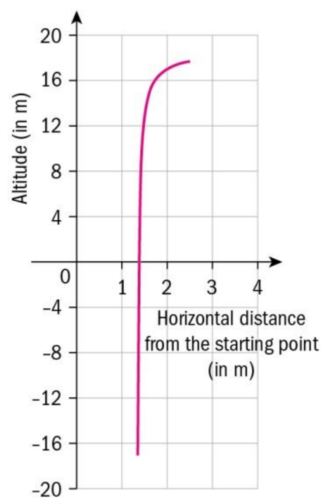
d


Altitude (in m)
20
16
12
8
4
0
1 2 3 4 5
Horizontal distance from the starting point (in m)

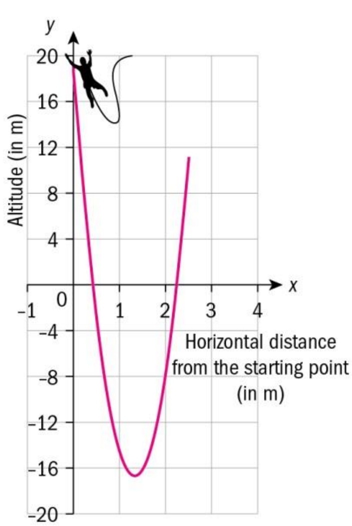
e


Altitude (in m)
20
16
12
8
4
0
1 2 3 4 5
Horizontal distance from the starting point (in m)

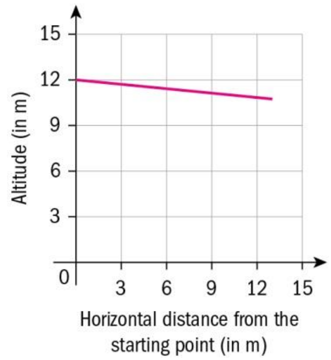
f


Altitude (in m)
20
16
12
8
4
0
1 2 3 4
-4
-8
-12
-16
-20
Horizontal distance from the starting point (in m)

g


y
Altitude (in m)
20
16
12
8
4
0
-1 1 2 3 4
x
-4
-8
-12
-16
-20
Horizontal distance from the starting point (in m)

h


Altitude (in m)
15
12
9
6
3
0
3 6 9 12 15
Horizontal distance from the starting point (in m)

a For each station:

i state the altitude of the visitor at the beginning of each station

ii state the altitude of the visitor when they are the furthest away from the starting point of the station

iv state the domain and range for each graph

iii state the altitude of the visitor when located at a horizontal position 1 m to the right of the start of the station.

b **i** Reorder the graphs to match the activities.

ii List the relations that are function(s).

c For each graph, state:

i whether it is possible to find precisely the horizontal distance of a visitor if their altitude is known

ii the functions that have an inverse function.

d The equations for the above graph are:

$R(x)_a = 5x + 3.8,\quad R(x)_b = |2x - 5| + 5,\ R(x)_c = logx\ x + 7,\ R(x)_d = 0.1(y - 14.5)^3 - (y - 12),$

$R(x)_e = -(x - 2.1)^3 + 6.5, R(x)_f = -\frac{1}{x-1.325} + 18.5, R(x)_g = 20x^2 - 53x + 18.5, R(x)_h = -0.1x + 12$

Use your graphing calculator to:

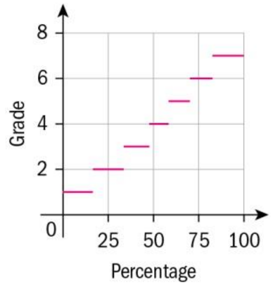
i Find the exact value of $R_\beta(1)$ with $\beta = a, b, c, d, e, f, g$ or h

ii estimate $R_\beta(x) < 10$

e Considering each of these relations one after each other as a piecewise function, state its equations and domains. https://www.desmos.com/calculator/57s4qebr0l

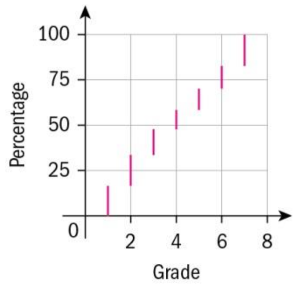
Answers

1 a i



https://www.desmos.com/calculator/b06tr7zari

ii



https://www.desmos.com/calculator/ck81bx3zkh

iii Yes

iv No

v **i** is a function but **ii** is not

vi i is false, ii Is true (in the same way that a DP student is always an IB student but an IB student is not always a DP student – they could be MYP or PYP).

vii i: range is $0 \le x \le 100$, domain is $\{1, 2, 3, 4, 5, 6, 7\}$

ii: domain is $0 \le x \le 100$, range is $\{1 ,2, 3, 4, 5, 6, 7\}$

viii i. $\begin{cases} y = 1 \text{ if } 0 < x \le 16 \\ y = 2 \text{ if } 16 < x \le 33 \\ y = 3 \text{ if } 33 < x \le 47 \\ y = 4 \text{ if } 47 < x \le 58 \\ y = 5 \text{ if } 58 < x \le 70 \\ y = 6 \text{ if } 70 < x \le 82 \\ y = 7 \text{ if } 82 < x \le 100 \end{cases}$ ii. $\begin{cases} x = 1 \text{ for } 0 < y \le 16 \\ x = 2 \text{ for } 16 < y \le 33 \\ x = 3 \text{ for } 33 < y \le 47 \\ x = 4 \text{ for } 47 < y \le 58 \\ x = 5 \text{ for } 58 < y \le 70 \\ x = 6 \text{ for } 70 < y \le 82 \\ x = 7 \text{ for } 82 < y \le 100 \end{cases}$

b i True

ii False

iii True

iv True

v False

2 For relations $i - iv$

a i domain is {1, 2, 3, 4, 5, 6, 7}; range is {1, 2, 3, 4, 5, 6, 7}

ii domain is {1, 2, 3, 4, 5, 6, 7}; range is {1, 4, 5, 6}

iii domain is {1, 2, 3, 4, 5, 6}; range is 2, 3, 4, 5, 6, 7}

iv domain is (2, 5, 6}; range is {3, 6, 7}

b i many to many as $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$

ii many to one as $(1,1)$, $(2,1)$

iii one to many as $(2,3)$, $(2,4)$

iv one to one

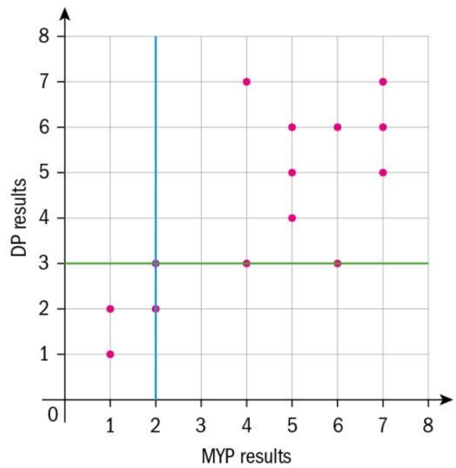
c i not a function

ii a function

iii not a function

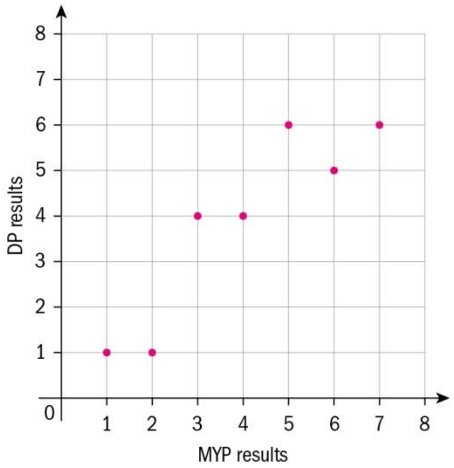
iv a function

d i



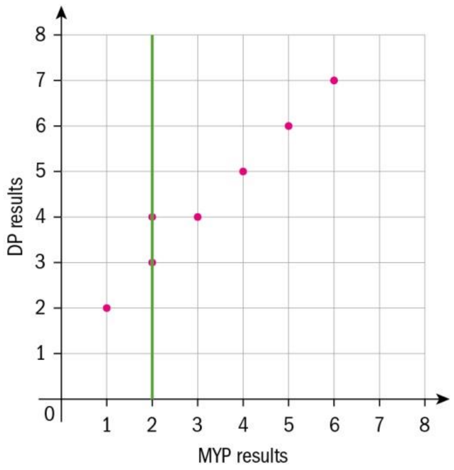
https://www.desmos.com/calculator/fck2apogyl

ii



https://www.desmos.com/calculator/zz11bdo0rb

iii



https://www.desmos.com/calculator/r5fv18f7cn

e It is not possible.

f i $R_i(1) = 1 \text{ or } 2$; $R_i(5) = 4,5, \text{or } 6$; $x = 7 \text{ or } 4$

ii $R_i(1) = 1$; $R_i(5) = 6$; $x = \emptyset$ 7 is not in the range

iii $R_i(1) = 2$; $R_i(5) = 6$; $x = 6$

3 a

a) i 0 ii.3.8 iii.Domain:[-0.76;0] range:[0, 3.8] iv. Never 1 m away

b) i. 10 ii.12 iii. Domain: [0, 6] range: [5, 12] , iv. 8

c) i. 3.8 ii. 8 iii. Domain: [0, 10] range: [3.8, 8], iv.7

d) i. 11 ii. 18 iii. Domain: [0, 4] range: [11, 18] , iv. 3 possible altitudes

e) i. 16, ii. 0 iii. Domain:[0;4] range:[0, 16] iv 1.8

f) i. -16.5 ii.16 iii. Domain: [0, 6] range: [-16.5, 16] iv. undefined

g) i. 18 ii. 11 iii. Domain: [0, 2.5] range: [-16.5, 18], iv.-14.5

h i. 12 ii. 11 iii. Domain: [0, 2.5] range: [-16.5, 18], iv.-14.5

b i a), c), b), h), d), g), f) and e)

ii They are all functions apart from (d

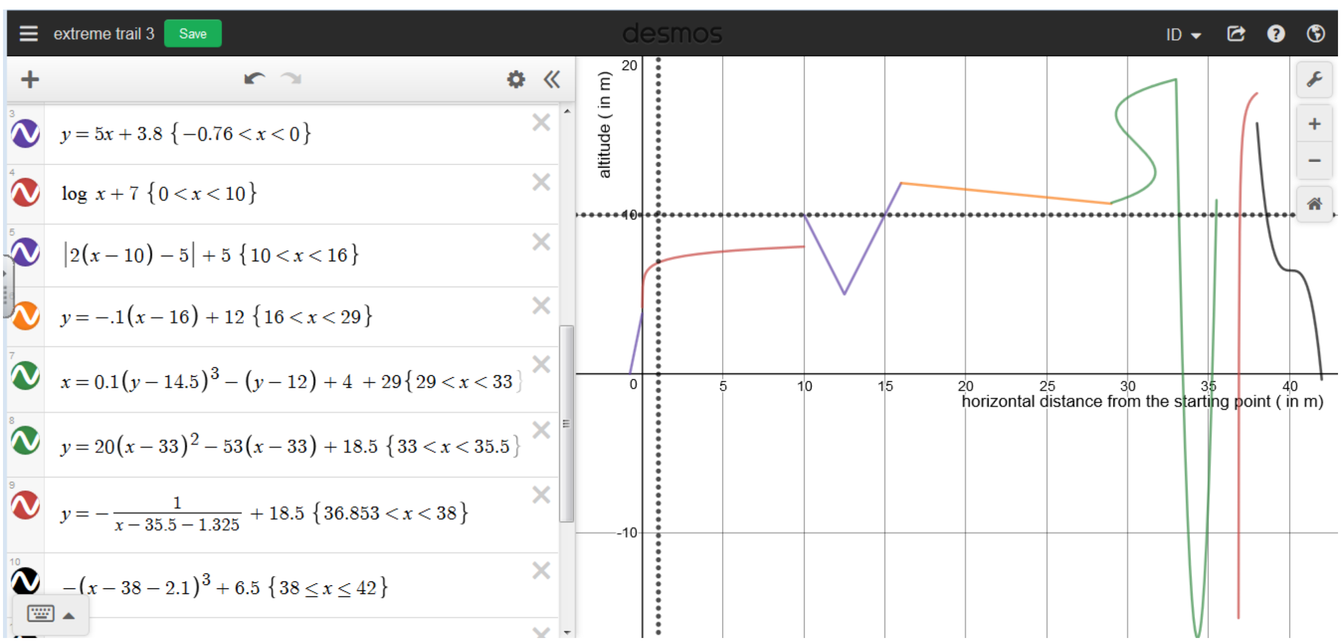
c i b) and g) are impossible. The rest are possible, although e) will require precision.

ii All but b) and g)

d i 4i..$R(x)_a = \emptyset$, $R(x)_b = 8$, $R(x)_c = 7$,$R(x)_d = 11.112, 15.014 \; or \; 17.314$, $R(x)_e = 7.931$, $R(x)_f = \emptyset$, $R(x)_g = -14.5$, $R(x)_h = 11.9$

ii all domain; b) $0 \le x \le 5$, c) all domain; d) $x = \emptyset$; e) $0.52 \le x \le 4$; f) $0.171 \le x \le 2.479$; g) $1.353 \le x \le 1.443$; h) $x = \emptyset$,

5.



https://www.desmos.com/calculator/2vec8n7wov

4.2 Linear models

1 Budgeting is one of the main preoccupations for university students. To help students with predicting the status of their bank account, universities provide them with cost and earning estimations for each of the following:

a Earnings:

i Tutoring opportunity GBP 20 per class, 8 classes a month.

ii Part-time job opportunity for 12 nights a month paying GBP 240 a month, plus an average of GBP 40 of tips a night.

b Spending:

i To rent a flat a student pays 3 months deposit and a monthly rent of GBP 600.

ii Food is on average GBP 265 a month.

iii A voucher of GBP 50 is given for the furniture at the beginning of the year. Then there is an average a monthly cost of GBP 10.

iv The gym membership is GBP 100 a year.

v A one-year telephone with internet package cost GBP 4.00 to open. It is free for the first 3 months and then costs GBP 25 a month for the rest of the contract.

2 For each of the above:

a find the independent variable and the corresponding unit

b find the dependent variable(s) and the corresponding unit

c find any constant and the corresponding unit

d find the rate of change (if it exists) and the corresponding unit

e determine whether a linear equation can be used to model the cumulative money earned or spent.

f determine whether the mapping will be positive or negative

g find an equation describing the relation in the form y in terms of x. where y represents the bank account balance, e.g. if it is spending, it should be negative, and x represents the number of months since the start of the course

h write down:

i its domain

ii its range

i find the initial charge/ gain

j sketch each relation

k state whether it is a function

l predict the amount of money spent/earned after

i 6 months

ii 1 year

m predict when the spendings/gains will add up to

i GBP 500

ii GBP 10,000 or higher

n write your answer as an inequality.

o find the inverse (if possible).

p write down in words what the inverse represents

q identify

i its domain

ii its range

r graph it

s explain whether this inverse is a function.

t Using all of the above **write** the mapping representing the money in a student's bank account at any time during the first year.

3 In August 2018, Ralf Van Hulle won the third edition of the Sun Trip, a 12,000-kilometers solar bike race aiming at promoting renewable energy. He completed the journey in 49 days.

a Identify:

i the independent variable and the corresponding unit

ii the dependent variable(s) and the corresponding unit

iii the constant.

b Calculate the rate of change and state its corresponding unit.

c Hence write down an equation describing the mapping.

d State

i its domain

ii its range

iii the properties of the relation.

4 If he had stopped one day for technical problem on the 28th day and his mean speed was the same every day:

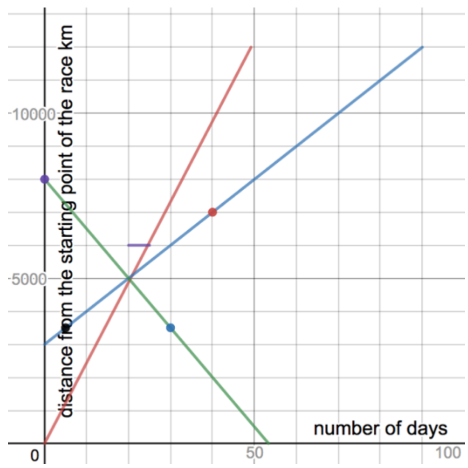
a sketch the graph

b state:

i the properties of the relation

ii the equation(s) mapping it and the corresponding domain(s).

5 The following graph represents Van Hulle's journey, the journey of people trying to complete part of the race and one spectator who waited for Ralf Van Hulle to pass by his house.



https://www.desmos.com/calculator/egfya3cwsh

a Identify:

i the graph representing the spectator

ii how long the spectator waited

iii how far the spectator's house is from the starting point of the race.

b Identify the other two cyclists and state:

i how far away from the starting point of the race they undertook their journey

ii their speed

iii the equation mapping their journey when they met Ralf Van Hulle

iv how many days it would take them to complete their fraction of the race.

Answers

1 a i a) the number of the class or the number of months doing tutoring

b) the overall money earned doing tutoring in GBP or the monthly money earned doing tutoring

c) 0

d) 160

e) a linear equation can be used

f) positive

g) $y = 160x$

h) i. $0 \leq x$ ii. $y \geq 0$

i) 0

j) https://www.desmos.com/calculator/2239z1qmne

k) function

l) i. 960   ii. -1920

m) i. after the first lesson of the 3rd month ii. The second lesson of the 62nd month (which would only happen if you do a PhD).

n) i. $x \geq 3.125$ ii. $x \geq 62.5$

o) $y = \frac{160}{x}$

p) The number of months required to earn a certain amount of money

q) i. $x \geq 160$ ii. $0 \leq y$

r) https://www.desmos.com/calculator/jlzqvwcoyf

s) Yes, it passes the vertical test

ii a) the number of months working

b) the overall money earned in GBP

c) 0

d) 720

e) a linear equation can be used

f) positive

g) $y = 720x$

h) i. $0 \leq x$ ii. $y \geq 0$

i) 0

j) https://www.desmos.com/calculator/l1ppvtxpkz

k) function

l) 4320 ii. -8640

m) i. after the 1st month; ii. The second lesson of the 14th month (which would only happen if you do a PhD).

n) i. $x \geq 1$; ii. $x \geq 14$

o) $y = \frac{720}{x}$

p) The number of months required to earn a certain amount of money

q) i. $x \geq 0$ ii. $0 \leq y$

r) https://www.desmos.com/calculator/p3lc8z5sqb

s) Yes, it passes the vertical test

b i a) the amount of time spent in the flat in a month

b) the overall rent in GBP

c) the deposit i.e. GBP 800

d) the monthly rent i.e. GBP 600

e) a linear equation can be used

f) negative

g) $y = -600x - 1800y$ is the overall rent after x months

h) i. $x \geq 0$ or $x\epsilon\mathbb{N}$ ii. $y \leq -1800$

i) -1800

j) https://www.desmos.com/calculator/taorje3ayw

k) function

l) -54000 ii. -90000

m) i. from the beginning ii. from the 14th month

n) i. $x \geq 0$; ii. $x \geq 14000$

o) $y = \frac{x + 1800}{600}$

p) The time spent in the flat according to the total amount of money spent on accommodation

q) i. $x \leq -1800$ ii. $y \geq 0 \; or \; y \in \mathbb{N}$

r) https://www.desmos.com/calculator/7eskoifjsq

s) Yes, it passes the vertical test

ii a) the number of months eating as a student

b) the overall cost of food in GBP

c) none

d) the monthly food spending i.e. GBP 265

e) a linear equation can be used

f) negative

g) $y = -265x$

h) i. $x \geq 0$ or $x \epsilon \mathbb{N}$ ii. $y \leq 0$

i) 0

j) https://www.desmos.com/calculator/a7ucvwgbgd

k) function

l) -1590 ii. -3180

m) i. from the second month ii. from the 38th month

n) i. $x \geq 2$; ii. $x \geq 38$

o) $y = -\frac{x}{265}$

p) The time spent eating as a student according to the total amount of money spent on food.

q) i. $x \leq 0$ ii. $y \geq 0 \; or \; y \in \mathbb{N}$

r) https://www.desmos.com/calculator/lnwvrtuta7

s) Yes, it passes the vertical test

iii a) the number of months as a student

b) the overall cost of furniture in GBP

c) 50

d) the monthly furniture spending i.e. GBP 10/month

e) a linear equation can be used

f) negative

g) $y = 50 - 10x$

h) i. $x \geq 0$ or $x \epsilon \mathbb{N}$ ii. $y \leq 50$

i) 50

j) https://www.desmos.com/calculator/s0pku4mtyz

k) function

l) -10 ii. -70

m) i. from the 55th month (i.e Masters level) ii. From the 1005th month, i.e not possible

n) i. $x \geq 55$; ii. $x \geq 1005$

o) $y = \frac{50-x}{10}$

p) The time spent as a student according to the total amount of money spent on furniture.

q) i. $x \ \leq 50$ ii. $y \geq 0 \ or \ y \ \epsilon \ \mathbb{N}$

r) https://www.desmos.com/calculator/ejphpddyzk

s) Yes, it passes the vertical test

iv a) the overall cost of the gym in GBP

b) there is no dependent variable as it is a constant

c) 100

d) none

e) a linear equation can be used

f) constant (flat)

g) $y = -100$

h) i. $0 \leq x \leq 12$ ii. $y = -100$

i) 100

j) https://www.desmos.com/calculator/dorxcd7wyj

k) function

l) -100 ii. -100

m) i. and ii. Both never

n) i. and ii. impossible mathematically – we would write $x\epsilon\emptyset$ for both o) $y = \frac{50-x}{10}$

p) the possible time as a member according to the membership fee

q) i. $x = \ -100$ ii. $0 \leq y \leq 12$

r) https://www.desmos.com/calculator/n7wbymilav

s) No, it does not pass the vertical test

v a) the number of months into the contract

b) the overall cost of the package in GBP

c) 4

d) the monthly fee after 3 months i.e. GBP 25/month

e) i. piecewise ii linear

f) constant followed by positive

g) $y = -4$ and $y = -4 - 25(x-3)$ or $y = -79 - 25x$

h) i. 0 $\leq x \leq 3$ ii. $y \leq -4$ and i. 3 $\leq x \leq 12$ ii. $y \leq -4$

i) **-**4

j) https://www.desmos.com/calculator/kww4nch12l

k) function

l) i. -79 ii. -229

m) i. never ii. never

n) impossible mathematically – we would write $x\epsilon\emptyset$ for both

o) x = -4 and $y = \frac{79+x}{25}$

p) The number of months since the start of the contract according to the total amount of money spent

q) i. $x = -4$ ii. 0 $\leq y \leq 3$ and i. $x \leq -4$ ii. 3 $\leq y \leq 12$

r) https://www.desmos.com/calculator/mh8eqjwbmw

s) No, it does not pass the vertical test at $x = -4$.

2 **t** When 0 $\leq x \leq 3$, $y = -1854 + 5x$, which is a linear function.

But when 3 $\leq x \leq 12$, $y = -1879 - 20x$

3 **a** **i** the number of days

Ii the number of kms completed

iii 0

b $\frac{12000}{49} = 244.89$ but we need to round down as $\frac{12000}{245}$=48.9 which is the end of the 48th day.

c $y = 244x$

d **i** $0 \le x \le 40$

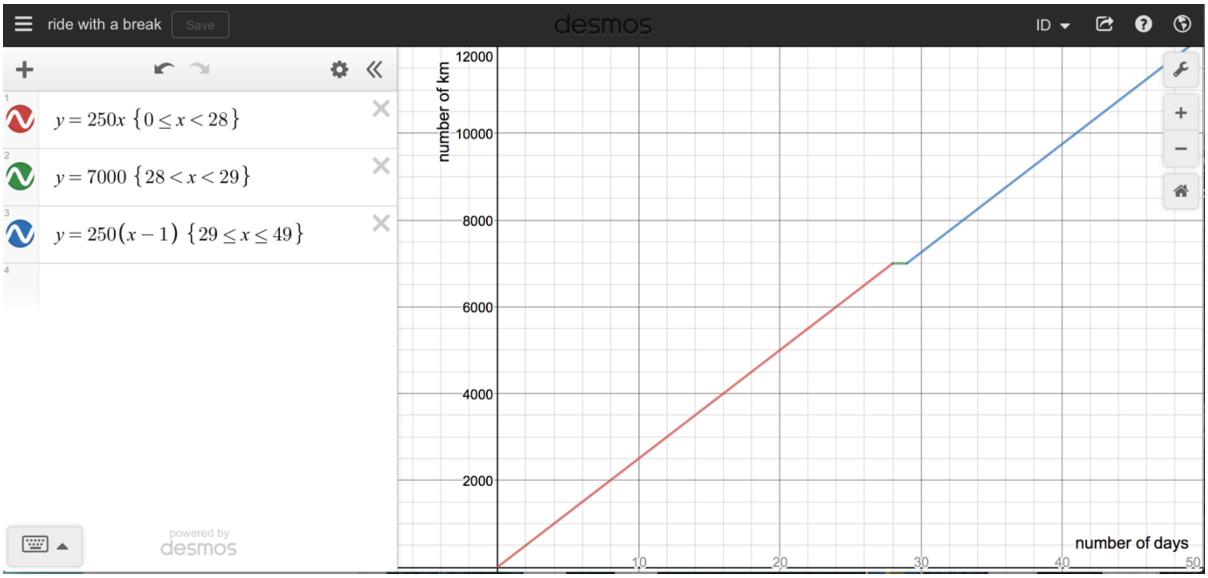
ii $0 \le y \le 12000$

iii **i**ncreasing linear function, direct variation

4 **a** https://www.desmos.com/calculator/4squnig2a4

b **i** piecewise relation

ii



5 **a** **i** purple

ii slightly less than 5 days

iii 6000km

5 **b**

	blue	green
i	3000 km	8000 km
ii	100 km/h	150 km/h
iii	$y = 100x + 3000$	$y = 8000 - 150x$
iv	99 days	53 days

4.3 Inverse functions

Saving our planet has been a focus for many governments and companies over the last few years.

A company plans to change their bulbs to LED. Currently, the lighting system costs $2 a month per neon tube, and there are 450 neon tubes equally used in the school. All the neon tubes have to be replaced every year. This costs $360 each time.

1 Find:

a the equation $E(x)$ for the total cost of the electricity after x months for $0 \leq x \leq 12$

b $E(x)$ as a composite function of 3 functions $f_a(x) \circ f_b(x) \circ f_c(x)$

c the 2 functions such that $f_a(x) \circ f_b(x) = f_b(x) \circ f_a(x)$

d the 2 functions such that $f_a(x) \circ f_b(x) \neq f_b(x) \circ f_a(x)$

e the value and meaning of a.$E(1)$, b. $E(2)$, c. $E(11)$ and d. $E(13)$

f when $E(x)$=3960 and hence $E^1(x)$

g the piecewise function $E(x)$ for $0 \leq x \leq 60$ if we consider the prices do not change.

2 State the range of $E(x)$ for $0 \leq x \leq 60$ and check your result graphing it.

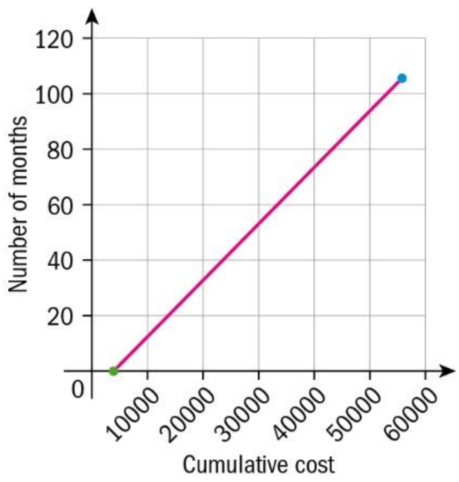
3 Explain whether:

a $E(x)$ for $0 \leq x \leq 60$ can be expressed as a composite function.

b $E(x)$ has an inverse.

4 State the domain and range of $E^{-1}(x)$ for $0 \leq x \leq 60$ and check your result by graphing it.

5 To compare the cost of the installations. The graph below shows the time a new LED installation can be used according to the money spent.



Find:

a the initial investment.

b the domain and range $T(x)$

c the number of month(s) of electricity one dollar will buy

d whether LED lamps will need to be replaced within the five years

e the equation of the of $T(x)$ in term of x

f the equation of the of $T(x)$ as a composite function $f_a(x) \circ f_b(x)$

g whether $f_a(x) \circ f_b(x) = f_b(x) \circ f_a(x)$

h $T^1(x)$ if it exists

i when the LED system becomes cheaper.

6 The company also convinces its staff to adopt more ecological behaviour. They use posters and regulations. Let x be the number of people doing at least on ecologically friendly action a day and $p(x)$ and $r(x)$ the number of people (still) doing daily eco-friendly actions after reading the posters and regulations respectively. A survey on their impact was done. Here are the results: $p = \{(0,5), (1,3), (2,4), (3,6), (4,5)\ (5,3), (6,4)\}$, and $r = \{(0,1), (1,4), (3,3), (4,3)\ (5,2), (6,3)\}$. Find:

a whether it is better to show the poster or the regulation first or if order does not matter
b the inverse function of $p(x)$ and $r(x)$ (if they exist)

7 The government plans to give tax incentive to make companies more ecologically friendly. They want to (1) give a \$1000 rebate and (2) give a 10% reduction for 'going green'.
Let x be the amount of tax before the incentive. Find:
a $a(x)$ and $b(x)$, the amount of money to pay after the \$1000 rebate and the 10% reduction, respectively
b whether one order is at the advantage of the companies
c whether $a(b(x))$ or $b(a(x))$ has an inverse function. If they do, state them and explain in words what they represent.

8 The government imposed the $b(a(x))$ order and added an extra 10% reduction to the company paying less than \$20000 of taxes before the incentive
a Find $(b \circ b \circ a)(x)$ and state the domain and range .
b Graph $(b \circ b \circ a)(x)$.
c Draw the inverse of $(b \circ b \circ a)(x)$.

Answers

1 **a** $E(x) = 360 + 900x$

b $E(x) = 360 + 450(2x) = f_3\left(f_2((f_1(x))\right)$ with $f_3(x) = 360 + x$, $f_2(x) = 450x$, and $f_1(x) = 2x$

c $f_2(x)$ and $f_1(x)$

d $f_3(x)$with any of $f_1(x)$

e **i** $E(0) = 360$, the initial cost

ii $E(2) = 2160$, the cost after 2 months

ii $E(11) = 10260$ the cost after 11 months and

ii $E(13) = \emptyset$ the function is only valid for 12 months so we cannot find the cost after 13 months

f when $x = 4$ and hence $T^1(x) = \frac{x-360}{900}$

g $E(x) = \begin{cases} 360 + 900x, & 0 \le x \le 12 \\ 720 + 900x, & 12 < x \le 24 \\ 1080 + 900x, & 24 < x \le 36 \\ 1440 + 900x, & 36 < x \le 48 \\ 1800 + 900x, & 48 < x \le 508 \end{cases}$

2 R= $y \in \{$ [360,11160] ∪]11520,22320] ∪]22680,33460] ∪]33820,44640] ∪]45000,55800]

https://www.desmos.com/calculator/xzuushoe6g

3 **a** No, as composite function cannot have piecewise domain.

b Yes, as the function is one to one.

4 D= $x \in \{$ [360,11160] ∪]11520,22320] ∪]22680,33460] ∪]33820,44640] ∪]45000,55800] and

R: $8 < y \le 50$

5 **a** 4000

b $4000 \le x \le 55800$ and $0 \le x \le 105.714$

c $\frac{1}{490}$

d No, as the function is continuous.

e $y = \frac{x}{490} - \frac{4000}{490}$

f $y = \frac{x-4000}{490} = f_a((f_b(x))$ with $f_a(x) = \frac{x}{490}$, and $f_b(x) = x - 4000$

g $f_b((f_a(x)) = \frac{x}{490} - 4000$ so its not equal

h $T(x)$is one to one so it has an inverse $T^{-1}(x) = 4000 + 490t$

i Graphing $T^{-1}(x)$ and $E(x)$ https://www.desmos.com/calculator/ujccbsd0jg x=8.8 so from the 8th month

6 **a** $p \circ r = \{(0,3), (1,5), (3,6), (4,6), (5,4), (6,6)\}$ and $r \circ p = \{(0,2), (1,3), (2,3), (3,3), (4,2)\ (5,3), (6,3)\}$, so it is better to show the regulations first.

b Both functions are many to one, so they do not have an inverse function.

7 **a** $a(x) = x - 1000$ and $b(x) = \frac{9}{10}x$

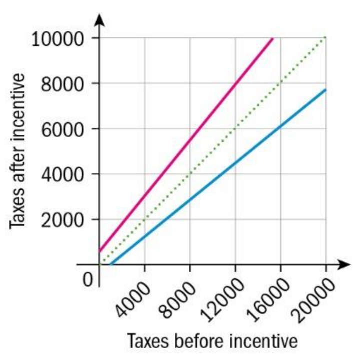
b $a(b(x)) = \frac{9}{10}x - 1000$ and $b(a(x)) = \frac{9}{10}(x - 1000) = \frac{9}{10}x - 900$, so first do the 10% reduction and then the \$1000 rebate.

c They both have inverse functions as they are both linear functions.

8 **a** $(a \circ b)^{-1}(x) = \frac{10x+10000}{9}$ and $(b \circ a)^{-1}(x) = \frac{10x}{9} + 1000$ the amount of taxes the company would pay if there were not green according to the amount of tax they pay after the incentive.

b $(b \circ b \circ a)(x) = = \frac{81}{100}x - 810,\ 1000 < x \leq 20\,000$. Note that $1000 < x$ as $(b \circ b \circ a)(1000) = 0$ and $0 < x \leq 20\,000$

c



https://www.desmos.com/calculator/rkcai7sgex

4.4 Arithmetic sequences and series

1 A new bakery shop has opened in Paris. The owner wants to increasingly develop her business. She started with 20 croissants on the first day. They plan to increase the number of croissants every day by 7. Let C_n be the number of croissants on the n^{th} day.

a Calculate:

i C_1

ii C_2

iii C_3.

b State:

i C_{n+1} in term of C_n.

ii C_n in term of n.

c Find the number of cakes produced:

i the 15^{th} day **ii** after a year.

d Calculate when:

i at least 1000 cakes are produced

ii they have multiplied their initial production by 10.

e Explain why producing 2000 croissants does not take twice as much time as producing 1000.

f Find whether there is any day the bakery produces exactly:

i 65 croissants

ii 650 croissants.

g Find how many croissants were produced:

i in a week

ii in 100 days

iii in a year.

2 The owner wants to proceed with the same business plan for her other products.

a If her production of baguettes follows the arithmetic sequence $299+b,\ 222, 60b+1$, find:

i b

ii the first term

iii the common difference

iv the formula for the general term.

b If her production of *pain au chocolat* follows the arithmetic sequence $p+3,\ 3p-2,\ 48$, find:

i p

ii the first term

iii the common difference

iv the formula for the general term.

c If she produced 103 macaroons on the 4^{th} day and 523 on the 18^{th}, find:

i the common difference

ii the first term

iii the formula for the general term.

d If she produced 103 macaroons on the 4^{th} day and 523 on the 18^{th}, find:

i the common difference

ii the first term

iii the formula for the general term.

3 For each of the above (**a** – **b**), find how much was produced:

i in a week

ii in 100 days

iii in a year.

4 As the business is going well the owner has decided to have a terrace with tables and chairs. If one table can seat 8 people, 2 tables can seat 12, 3 tables can seat 16.

a Sketch a diagram to visualise the problem.

b Find how many people can sit at

i 4 tables

ii 5 tables

iii 10 tables.

c State the equation mapping the number of tables, t, to the number of people, p.

d Calculate how many tables are needed if a group of 60 people comes.

e If new tables are bought and we can sit n people on one table, $2(n\text{-}1)$ people on two tables and $4n$ people on three tables, find:

i n

ii the common difference

iii the formula for the general term.

f The owners decides to decorate her tables with flowers. She puts 1 flower on the first table, 2 flowers on the second table, 3 flowers on the third. Find:

i the common difference

ii the general formula for the number of flowers on the nth table.

iii the formula for the for the total number of flowers she needs to buy for n tables.

5 To build her bakery the owner borrowed money.

a She borrowed 1500 euros at 0.1% simple monthly interest for 2 years. Find:

i how much interest she will pay (a) after one month; (b) after 2 years

ii the monthly repayment

iii the general formula to model the amount of money H she has paid after n months.

b She also borrowed from another friend at 0% interest. She calculated she still had to reimburse $19x$ euros after the first repayment, $21x - 150$ after the second, $11x + 250$ after the third. Find:

i x

ii the monthly repayment

iii the amount borrowed

iv the general formula to model the amount she still owes A

v how long it will take her to reimburse.

c She borrowed 6000 euros from the bank. Her monthly repayments are 3% of the amount borrowed. She will have to pay them for 3 years. Find:

i the monthly repayment

ii the general formula to calculate how much she has paid back after n months

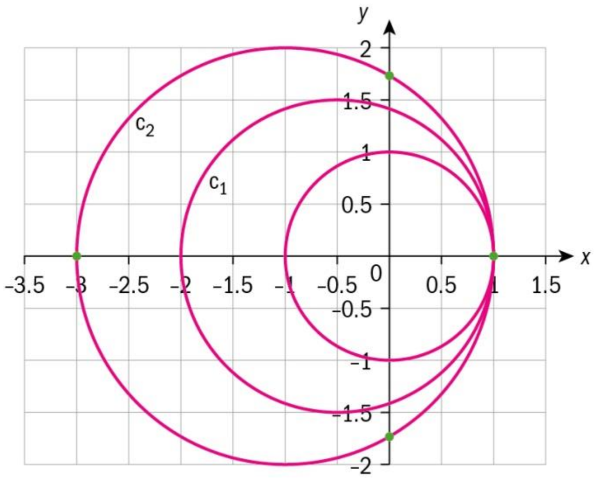
iii how much she will have paid back to the bank at the end of her 3 years

iv the total charge from the bank

v the equivalent annual simple interest charged by the bank

vi the equivalent monthly simple interest charged by the bank.

6 The logo will be made of a series of circles outside each other as shown below. The croissant area C_1,... C_n will be painted in alternating colours.



a Set r_n as the radius of each circle. With $r_0=1$, find:

i r_1

ii r_2

iii r_3

b Write down a formula to represent r_n:

i in terms of r_{n-1}

ii in terms of n.

c Calculate:

i C_1

ii C_2

iii C_3.

d State:

i C_{n+1} in term of C_n

ii C_n in term of n.

e Find C_{10}.

f Find n such that $C_n<10$.

g Explain whether C_n can be an integer.

7 Calculate the surface area of

a **i** $C_1 + C_3$

ii $C_1 + C_3 + C_5$

b **i** $C_2 + C_4$

ii $C_2 + C_4 + \ldots + C_{8.}$

8 For (a) $C_1 + C_3 + \ldots C_{2k+1}$ with n an odd number, and (b) $C_2 + C_4 + \ldots C_{2k}$ with n an even number:

i write the sum using the sigma notation

ii write the general formula to model the sum in terms of n.

Answers

1 a i 20

ii 27

iii 34

b i $C_{n+1} = C_n + 7$

ii $C_n = 7n+13$

c i 118

ii 2568

d i 141 days

ii 284 days

e Because on the first day there were 20 croissants so 13 croissants extra (hence nearly 2 days' worth of extra croissants)

f i impossible: on the 7th day 62 croissants are made and on the 8th day 75 are made

ii possible: it happens on the 91st day

g i 287

ii 713

iii 2568

2 a i b = 4 **ii** 305 **iii** 19 **iv** 289 + 19b

b i p = 11 **ii** 14 **iii** 17 **iv** 17p - 3

c i 30 **ii** 13 **iii** 30m - 17

d i 11 **ii** 18 **iii** 11w + 7

3 a i 419 **ii** 2186 **iii** 7221

b i 116 **ii** 1697 **iii** 6202

c i 193 **ii** 2983 **iii** 10933

d i 84 **ii** 1107 **iii** 4022

4 b i 20 **ii** 24 **iii** 44

c p = 4t + 4

d 14

e i $n = 8$ **ii** 6 **iii** $6n + 2$

f i 1 **ii** n **iii** (n + 1)/2

5 a i a) 1.5; b)36

ii 64

iii H = 64n

b **i** 50 euros

ii 950 euros

iii 1000 euros

iv A = 1000-50n

v 20 repayments

c **i** 180 euros

ii 180n

iii 6480 euros

iv 480 euros

v 2.7%

vi 0.2%

6 **a** **i** 1.5 **ii** 2 **iii** 2.5

b **i** $r_n = r_{n-1} + 0.5$ **ii** $r_n = 1 + 0.5n$

c **i** 1.25π **ii** 1.75π **iii** 2.25π

d **i** $C_{n+1} = C_n + 0.5\pi$ **ii** i.$C_n = (0.75 + 0.5n)\pi$

e 5.75π

f 4

g Impossible as π is an irrational number.

7 **a** **i** 3.5π

ii 6.75π

b **i** 4.5

ii 20.5

8 **a** **i** $\sum_{k=1}^{2k-1} C_k$ **ii** $\sum_{k=1}^{2k} C_k$

b **i** $\sum_{k=1}^{2k-1} C_k = \frac{k}{2}(1.5 + k)$ **ii** $\sum_{k=1}^{2k} C_k = = \frac{k}{2}(2.5 + k)$

4.5 Linear regression

Cinderella and Prince Charming lived happy ever after… But what about the reality? Below are the annual percentages of risk of break up, by years of relationship:

Year(s) together	1	2	3	4	5	8	10	15	20	22	25	28	29	36	40
Annual rate for married couples as a %	4	4	3.5	3.6	3.4	3.2	3.1	3	2	2.2	2	1.8	1.7	1.5	1.2
Annual rate for non-married couples	70	32.5	25	22	20	16	14	12	11	10.8	11	11.3	11.5	13	14.5

1 Analyse the correlation between the rate of annual break up and the number of years together for married couples. For this you should:

- **a** plot a scatter diagram for these data
- **b** calculate the Pearson product moment correlation coefficient for these data
- **c** describe the correlation
- **d** write down the equation of the regression line of y on x
- **e** plot and label the mean point on the scatter
- **f** draw the line of best fit
- **g** predict the chance of breaking up after 30 years
- **h** state the number of years after which the chance of break up is zero.

2 Analyse the correlation between the rate of annual break up and the number of years together for unmarried couples. For this you should repeat **a** to **g**.

3 Explain how using a piecewise function would be better to map the correlation for non-married couples, state the equation and the domain.

4 Tyler Vigen in *Spurious Correlations* analysed the correlation between the consumption of margarine per capita and the divorce rate in Maine. Use the following data to:

Margarine consumed (in lbs)	8.2	7	6.5	5.3	5.2	4	4.6	4.5	4.2	3.7
Divorce rate in Maine (in %)	5	4.7	4.6	4.4	4.3	4.1	4.2	4.2	4.2	4.1

a plot a scatter diagram for these data

b calculate the Pearson product moment correlation coefficient for these data.

c describe the correlation

d write down the equation of the regression line of y on x.

e find the mean quantity of margarine consumed

f find the mean of the percentage of divorce

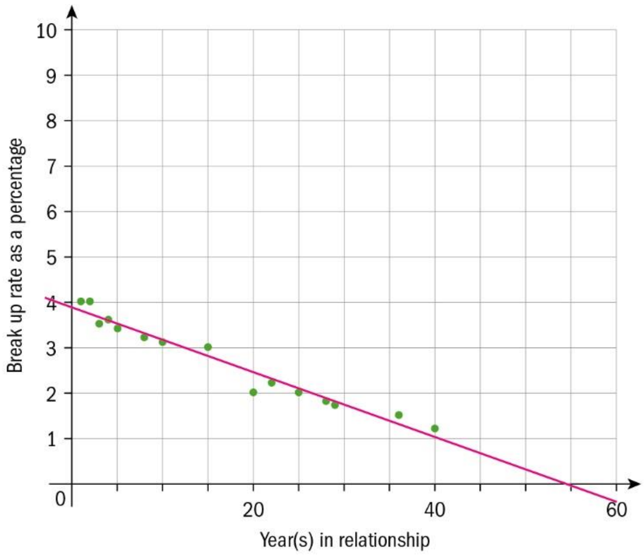
g plot and label the mean point on the scatter

h draw the line of best fit

i discuss whether you can predict the percentage of divorce of a couple who eats 9lbs of margarine.

Answers

1 **a** https://www.desmos.com/calculator/mgxybpfgp2



b -0.98.

c Strong negative linear correlation

d See above

e The mean is the point (16.53,2.68)

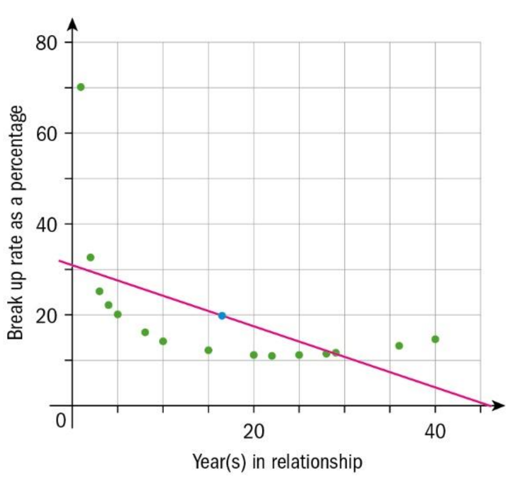
f See above

g 1.7%

h 54 years

2 a

https://www.desmos.com/calculator/fprx2tfmn4



b -0.671 therefore moderate negative correlation

c $y = . -0.67x + 30.7$

d See above mean point: (16.53,19.64)

e See above

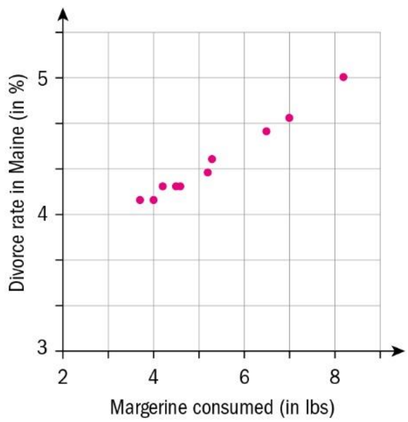
f 10.6%

g This mapping seems to be made of 3 parts: a dramatic drop for the first three years, a slow drop till 15years and then the percentage of break up rise again slightly.

3 https://www.desmos.com/calculator/khnm9lcm05

4 **a**

https://www.desmos.com/calculator/toc9p76pjw



b r =99.26%

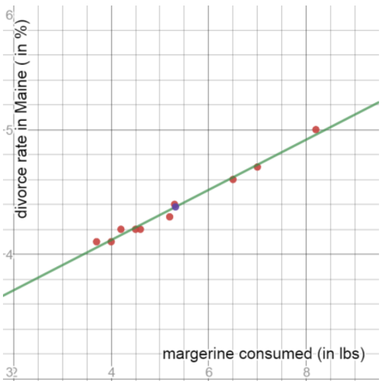
c very strong correlation

d $y = 0.2x + 3.3$

e 5.32 lbs

f 4.8%

g https://www.desmos.com/calculator/toc9p76pjw



h See above

I According to the graph if x=9. Y=5.121%, this is however a spurious correlation. Those two variables are clearly independent and the correlation cause is due to another factor or mere chance.

For more spurious correlations, visit http://www.tylervigen.com/spurious-correlations

5.1 Theoretical and experimental probability

1 A letter is picket at random from the word PROBABILITY. What is the probability that it is a letter from the word FACTS?

2 A bag contains 12 counters numbered from 1 to 12. If a counter is picked at random, find the probability that it is:

a even **b** a cube number **c** prime and odd

3 A survey of the most common method of grocery shopping was carried out amongst a group of adults. The results are shown in the following table:

Method	**Online**	**Instore**
Age 20-39	64	31
Age 30-59	53	49
Age 60 and over	27	81

One adult is selected at random from the group.

a What is the probability that this adult shopped online?

b If this sample is representative of the general population of adults aged 20 and over, how many people would you expect to shop instore in a town containing 80 000 adults aged 20 and over?

4 A fair icosahedral die (numbered 1 to 20) is rolled. What is the probability that it will show:

a a factor of 20 **b** a prime number?

5 A baseball team wins 60% of its home fixtures and 30% of its away fixtures. In a season where it plays 18 home matches and 15 away matches, how many matches would it be expected to win?

Answers

1 $\frac{2}{11}$

2 **a** $\frac{6}{12} = \frac{1}{2}$ **b** $\frac{2}{12} = \frac{1}{6}$ **c** $\frac{4}{12} = \frac{1}{3}$

3 **a** $\frac{144}{305}$

b $\frac{161}{305} \times 80000 = 42{,}229.5\ldots \approx 42{,}230$

4 **a** $\frac{6}{20} = \frac{3}{10}$ **b** $\frac{8}{20} = \frac{2}{5}$

5 $0.6 \times 18 + 0.3 \times 15 = 15.3$, so expect them to win 15 matches.

5.2 Representing combined probabilities with diagrams

1 In a school year group of 120 students, 53 study Spanish and 68 study French while 20 study neither French nor Spanish.

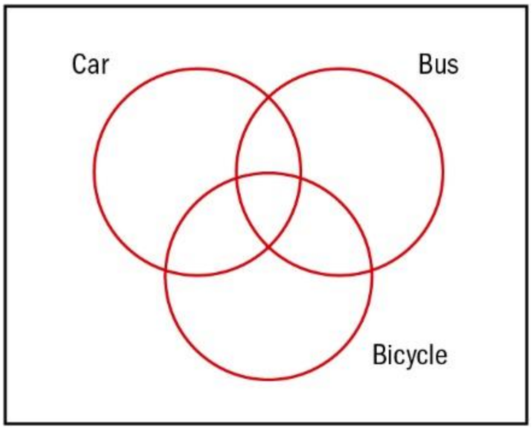
a How many study both French and Spanish?

b What is the probability that a student chosen at random studies Spanish but not French?

2 In a survey, 120 people were asked about their use of cars, buses and bicycles. It was found that 50 people used cars, 44 used buses and 46 used bicycles. Also 20 people used both cars and buses, 11 used both buses and bicycles,17 used bicycles only and

7 used all 3 modes of transport.

a Copy this Venn diagram with numbers assigned in all regions.



b What is the probability that someone selected at random from this group does not use any of these three modes of transport?

3 Two fair spinners are designed. One is numbered from 1 to 5 while the other is numbered from 1 to 4. Each is spun and the product X of the two numbers obtained is calculated. Find:

a P(X is even)

b P(X is a square number)

c P(X is a multiple of 3).

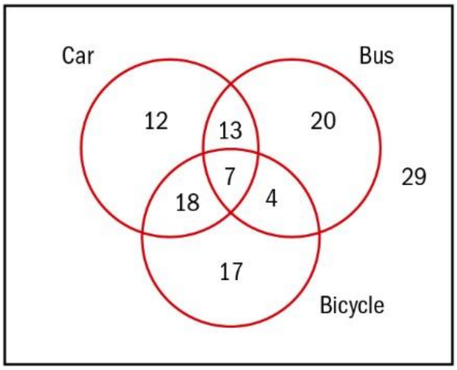
4 Five fair coins are tossed. What is the probability of getting three heads and two tails?

5 Two fair dice are rolled. What is the probability that the difference between the two numbers is greater than two?

Answers

1 **a** 21 **b** $\frac{32}{120} = \frac{4}{15}$

2 **a**



b $\frac{29}{120}$

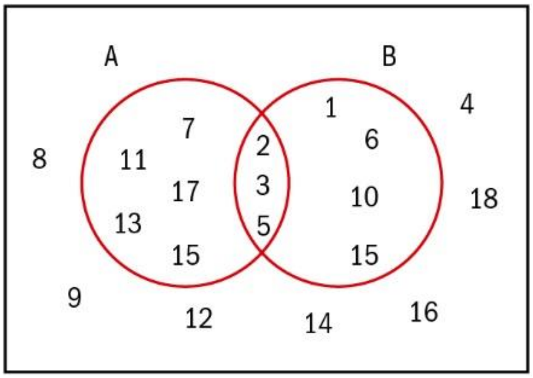
3 **a** $\frac{7}{10}$ **b** $\frac{3}{10}$ **c** $\frac{2}{5}$

4 $\frac{10}{32} = \frac{5}{16}$

5 $\frac{12}{36} = \frac{1}{3}$

5.3 Representing combined probabilities with diagrams and formulae

1 A fair icosahedral (20-sided) die numbered 1,2,3, … 20 is rolled and the number noted. The events A 'roll a prime number' and B 'roll a factor of 30' are represented in the Venn diagram below.



a Find $P(A), P(B), P(A \cap B)$ and $P(A \cup B)$.

b Hence show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

2 For these pairs of events, state whether they are mutually exclusive, independent or neither:

a A - It will be cloudy today.

B – It will be sunny tomorrow.

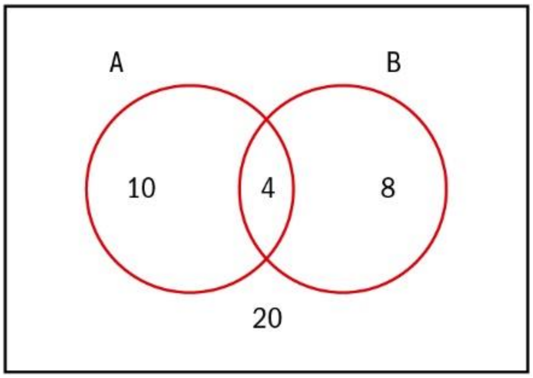
b C – My baseball team will win their next game.

D - My baseball team will lose their next game.

c E – I will roll a six on a fair die.

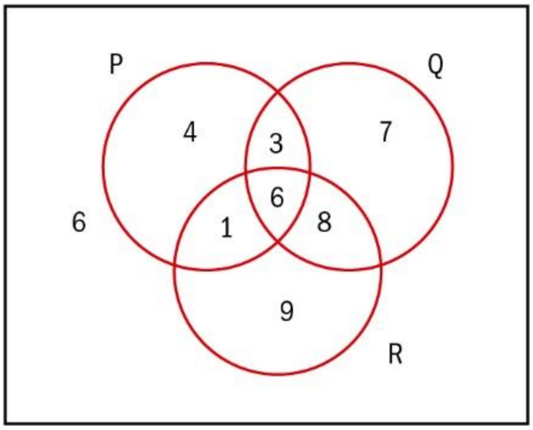
F – I will throw a tail on a fair coin.

3 In a group of 42 students in a school, A is the set of students studying art and B is the set of students studying biology. The numbers are shown in the Venn diagram below:



Show that A and B are independent.

4 The numbers of occurrences of the events P, Q and R are shown in the following Venn diagram.



Find:

a $P(P|Q)$

b $P(Q|R)$

Answers

1 **a** $P(A)=\frac{2}{5}, P(B)=\frac{7}{20}, P(A\cap B)=\frac{3}{20}, P(A\cup B)=\frac{3}{5}$

b $P(A)+P(B)-P(A\cap B)=\frac{2}{5}+\frac{7}{20}-\frac{3}{20}=\frac{3}{5}=P(A\cup B)$

2 **a** Neither

b Mutually exclusive

c Independent

3 $P(A)=\frac{14}{42}=\frac{1}{3}. P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{4}{42}}{\frac{12}{42}}=\frac{1}{3}=P(A)$ so independent.

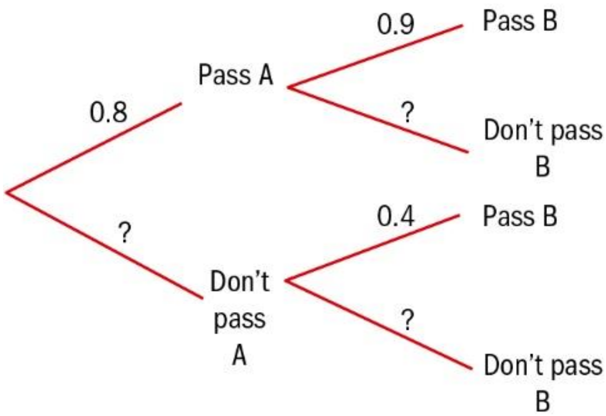
4 **a** $\frac{P(P\cap Q)}{P(Q)}=\frac{\frac{9}{44}}{\frac{24}{44}}=\frac{3}{8}$

b $\frac{P(Q\cap R)}{P(R)}=\frac{\frac{14}{44}}{\frac{24}{44}}=\frac{7}{12}$

5.4 Complete, concise and consistent representations

1 Betty is about to take two mathematics tests: A and B. The probability that she passes test A is 0.8. If she passes this test then the probability that she will pass test B is 0.9 but, if she fails test A, her confidence drops and the probability that she passes test B is 0.4.

a Copy and complete the tree diagram below



b Calculate the probability that she will pass test B.

2 A bag contains 10 red counters, 6 blue counters and 4 green counters. A counter is taken out at random and then replaced. This is repeated a second time. Find the probability that the two counters selected were the same colour.

3 A town has three districts: Alphaville, Betaville and Gammaville. 50% of the town live in Alphaville, 30% in Betaville and the remainder in Gammaville. A virus is spreading through the town so that, at present, 3% of Alphaville are infected, 2% of Betaville are infected and 1.5% of Gammaville are infected.

a What percentage of the town are infected?

b An inhabitant is selected at random and found to be infected. What is the probability that they come from Alphaville?

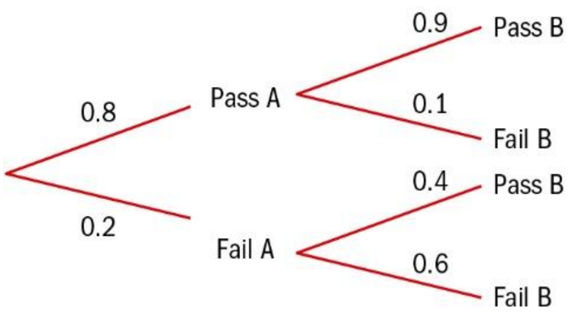
4 A bag contains 10 red counters and 10 blue counters.

a If three counters are taken out at random with replacement, what is the probability that at least on counter is red?

b Repeat part a without replacement.

Answers

1 **a**



b $0.8 \times 0.9 + 0.2 \times 0.4 = 0.8$

2 $\frac{1}{2} \times \frac{1}{2} + \frac{3}{10} \times \frac{3}{10} + \frac{1}{5} \times \frac{1}{5} = \frac{19}{50}$

3 **a** $0.5 \times 0.03 + 0.3 \times 0.02 + 0.2 \times 0.015 = 0.024$. So 2.4% of the town is infected.

b $P(A \mid V) = \frac{P(A \cap V)}{P(V)} = \frac{0.5 \times 0.03}{0.024} = 0.625$

4 **a** $1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$ **b** $1 - \frac{1}{2} \times \frac{9}{19} \times \frac{4}{9} = \frac{17}{19}$

6.1 Quadratic models

1 **a** Use technology to sketch the following quadratics and transfer the graph onto paper:

i $f(x) = 3x^2 + 2x - 5$ for $-5 \leq x \leq 5$.

ii $f(x) = -2x^2 - 5x + 8$ for $-5 \leq x \leq 3$

b For each of these graphs, find:

i the coordinates of the point where the graph crosses the y-axis

ii the coordinates of the zeros

iii the coordinates of the vertex

iv the equation of the axis of symmetry.

c State the range for the given domain.

d State the range if the domain was unrestricted.

2 A rectangular playing field has a perimeter of 110 metres.

a If the length of the field is x metres, find an expression, in term of x, for the width of the field.

b Find an equation for the area, A m^2, of the field.

c Using a suitable domain and range, sketch the graph of your equation from **b**.

d Find the x-intercepts and interpret them.

e Find the equation of the line of symmetry.

f Find the maximum area of the field and the value for x where this occurs.

3 Polly throws a beanbag into the air. The path of the beanbag can be modelled by the function $f(t) = -2.5t^2 + 10t$, where t is the time in seconds and $f(t)$ is the height of the beanbag in metres:

a Use technology to draw the graph of this function and transfer the graph onto paper.

b Find the intercepts with the x-axis and explain what these values mean.

c Find the equation of the axis of symmetry.

d Find the maximum height reached by the beanbag and the time when this occurred.

e Find the times when the beanbag reaches a height of 6 metres.

4 A bridge over a stream can be modelled by the function $f(x) = -0.02x^2 + 0.4x + 2$, where $f(x)$ represents the vertical height in metres and x the horizontal distance in metres.

a Sketch a graph of this function.

b Find the coordinates of the vertex.

c Find the coordinates of the points where the graph cuts the x-axis and explain what these points represent.

5 Sketch the graphs of the following functions on paper, clearly showing the x- and y-intercepts and the vertex.

a $f(x) = 3(x-1)(x+5)$

b $g(x) = -2(x-3)(x+8)$

c $h(x) = (3x-1)(2x+9)$

6 The average temperature in Breda, in °C, each month for one year can be modelled by the function $T(x) = -0.64x^2 + 7.67x - 2$ where $T(x)$ represents the temperature and x represents the months starting with January = 0.

a Graph this equation for $0 \le x \le 12$.

b Find the points where the graph crosses the x-axis and explain what these values represent.

c Find the maximum temperature and the month when this occurs.

d Cars should have their winter tyres put on when the temperature falls below 8°C. Find which months cars should have winter tyres.

7 The side view of a large pothole in a country road can be modelled by the function

$f(x) = 0.018x^2 - 0.53x$, where x is the width in cm and $f(x)$ is the depth in cm.

a Use a suitable domain and range to sketch the graph of this function.

b Find the coordinates of the vertex and explain what this represents.

c Write down the equation of the axis of symmetry.

8 For each of the following functions:

$f(x) = (x-4)^2$

$f(x) = 2(x+1)^2 - 3$

$f(x) = 5 - (x-1)^2$

a explain why $f^{-1}(x)$ does not exist if the domain of $f(x)$ is $x \in \Re$

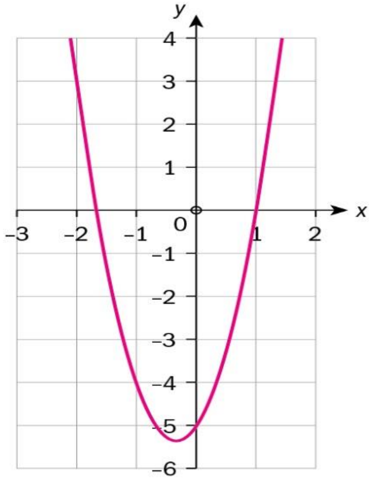
b find the largest possible domain for the function to have an inverse

c find this inverse

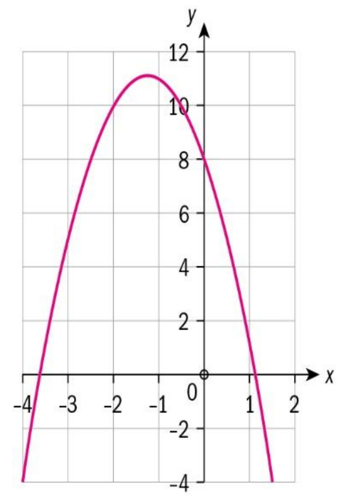
d using the restricted domain, sketch the graph of the function and its inverse on the same diagram.

Answers

1 a i



ii

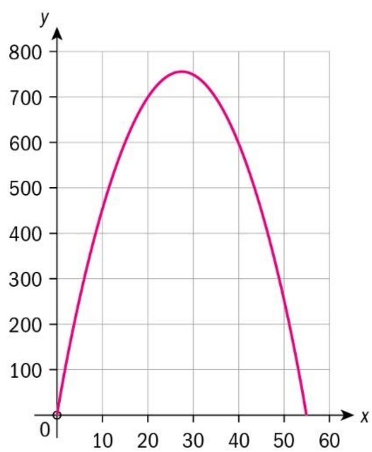


		i	ii
b	**i**	(0, -5)	(0, 8)
	ii	(-1.67, 0) and (1, 0)	(-3.61, 0) and (1.11, 0)
	iii	(-0.333, -5.33)	(-1.25, 11.125)
	iv	x = -0.333	x = -1.25
	v	$-5.33 \leq f(x) \leq 80$	$-25 \leq f(x) \leq 11.125$
	vi	$f(x) \geq -5.33$	$f(x) \leq 11.125$

2 a 55 – x

b $A = 55x - x^2$

c

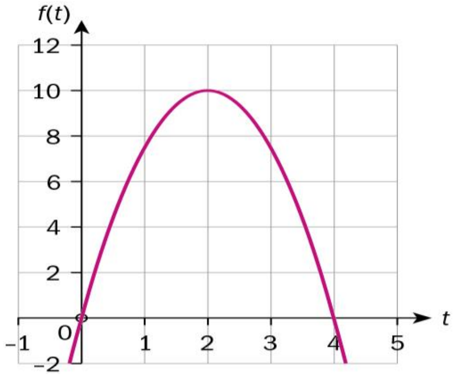


d (0, 0) and (55, 0) The value of x must be greater than 0 and less than 55.

e x = 27.5

f 756.25 m^2 when x = 27.5 m

3 **a**



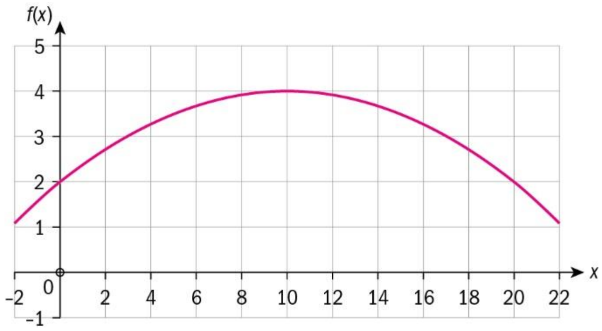
b (0, 0) and (4, 0). 0 seconds is when the beanbag is thrown and 4 seconds is when it returns to the starting place.

c x = 2

d 10 metres when x = 2 seconds.

e 0.735 seconds and 3.26 seconds

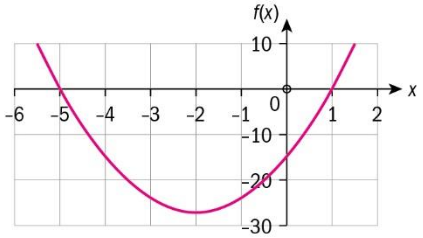
4 **a**



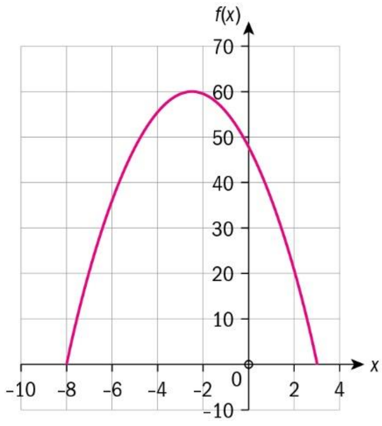
b (10, 4)

c -4.14 and 24.14 These are where the bridge starts and ends.

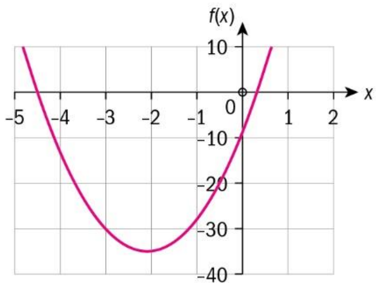
5 **a**



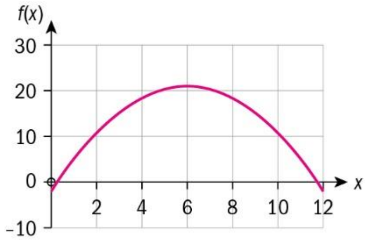
b



c



6 a

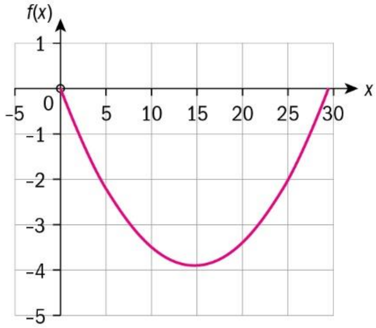


b 0.267 and 11.7 These are the times when the temperature is 0°C.

c 21°C when x = 6, so July.

d x = 1.49 and 10.5, so they need winter tyres approximately between November and February.

7 a



b (14.7, -3.9) This is the deepest point of the pot hole

c x = 14.7

8 a The inverse does not exist when the domain is all real numbers because the functions are one-to-many.

b i $x \geq 4$

ii $x \geq -1$

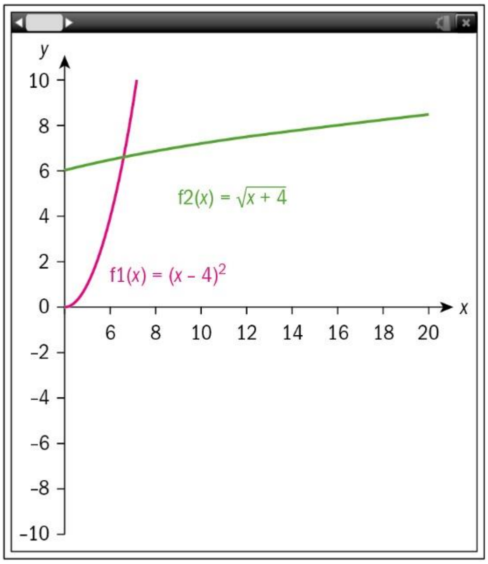
iii $x \leq 1$

c i $f^{-1}(x) = \sqrt{x} + 4$

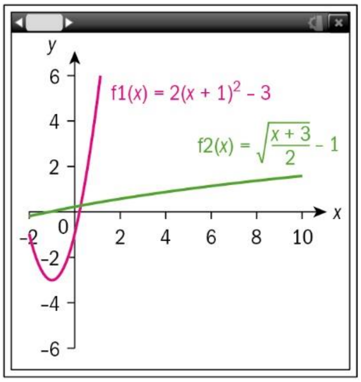
ii $f^{-1}(x) = \sqrt{\frac{x+3}{2}} - 1$

iii $f^{-1}(x) = \sqrt{(5-x)} + 1$

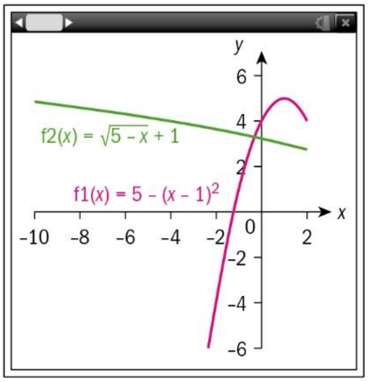
8 d i



ii



iii



6.2 Quadratic modelling

1 Aron dives from the top springboard into a swimming pool. The springboard is 10 metres above the edge of the pool. Aron's dive can be modelled by the function $f(t) = -2t^2 + 4t + 10$, where t is the time in seconds.

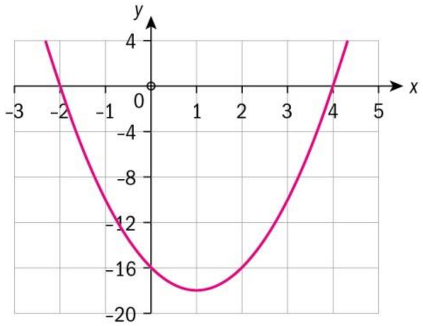
a Sketch the graph of Aron's dive.

b Find the maximum height reached.

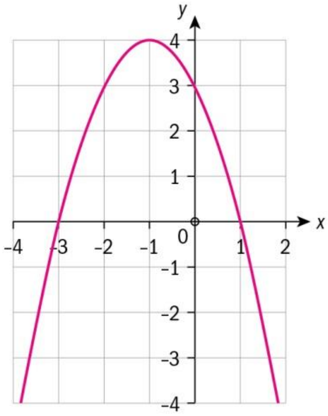
c Find the time when Aron hits the water.

2 Find the equations of the following graphs:

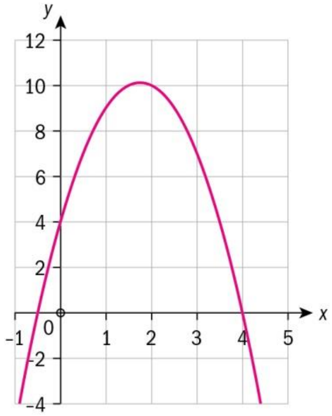
a



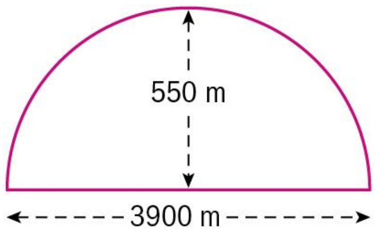
b



c



3 Lupin bridge has a total length of 3900 metres and a span of 550 metres as shown on the diagram.



Find a quadratic equation to model this bridge.

4 A quadratic function has equation $f(x) = ax^2 + bx + c$.

The points (1, -15), (2, 0) and (3, 21) all lie on the graph of the function.

Find the values of *a*, *b* and *c*.

5 The value of Nasakle's shares was recorded over a period of 24 months.

Month	2	4	6	8	10	12	14	16	18	20	22	24
Value in GHS	55	58	59	63	66	68	67	65	60	56	55	54

a Using your GDC or technology, plot these points. Put the month on the *x*-axis and the value on the *y*-axis.

b Using your GDC or technology, find the best fit quadratic equation through these points.

c Is this a good fit?

d Can you use your equation to estimate the value of her shares in month 9?

e Can you use this equation to predict what the value of the shares will be at a particular time in the future?

6 The number of votes that a politician can expect in the election was monitored over a period of 15 weeks. The results are shown in the table.

Week number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of votes	6200	6000	5700	5600	5400	5250	5200	5250	5350	5550	5600	5750	5800	6000	6100

a Using your GDC or technology, find the best fit quadratic equation through these points.

a State whether or not the equation is a good fit for the data and justify your answer.

7 Give a full geometrical description of the following transformations:

a From $f(x) = x^2$ to $g(x) = 2x^2 - 1$.

b From $f(x) = (x+2)^2 - 1$ to $g(x) = x^2 + 2$.

8 The graph of $f(x) = x^2$ is transformed into the graph of $g(x)$ by the following sequence of transformations:

A horizontal translation of 2 units to the right, followed by

A vertical reflection in the x-axis, followed by

A vertical stretch with scale factor 3.

Find $g(x)$.

9 Consider the function $f(x) = x^2$ and the transformations $r(x) = x - 2$ and $s(x) = 3x$

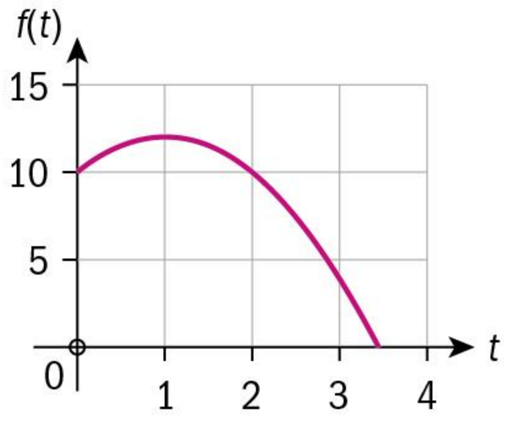
Find the following composite functions and give a full description of the transformations.

a $(s \circ f \circ r)(x)$

b $(r \circ f \circ s)(x)$

Answers

1 a



b 12 metres

c 3.45 seconds

2 a $f(x) = 2x^2 - 4x - 16$

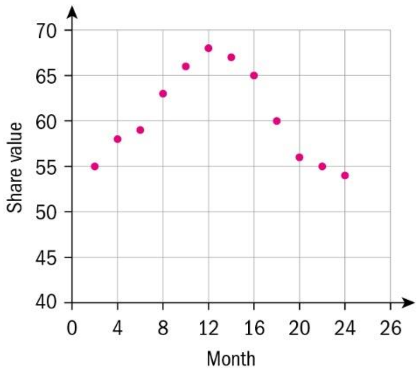
b $f(x) = -x^2 - 2x + 3$

c $f(x) = -2x^2 + 7x + 4$

3 $f(x) = -0.000145x^2 + 0.564x$

4 a = 3, b = 6 and c = -24

5 a



b $f(x) = -0.105x^2 + 2.59x + 49.5$

c The coefficient of determination, $r^2 = 0.85575$, shows that this model is a good fit.

Also, r = 0.925 which is a strong correlation.

d Yes.

e You could use it to predict values any time during the 24 months given, but not afterwards. This would be an example of extrapolation.

6 a $f(x) = 18.1x^2 - 283x + 6419$

b $r^2 = 0.921$ and $r = 0.960$. Since the coefficient of determination is almost a perfect fit and the correlation coefficient is very strong, the model is a very good fit.

7 **a** Vertical stretch with scale factor 2 followed by vertical translation of 1 unit down with translation vector $\binom{0}{-1}$

b Horizontal translation of 2 units to the right, vector $\binom{2}{0}$ followed by vertical translation of 3 units up with translation vector $\binom{0}{3}$

8 $g(x) = -3(x-2)^2$

9 **a** $3(x-2)^2$

b $9(x-2)^2$

6.3 Cubic functions and models

1 **a** Sketch the graphs of the following cubic functions and transfer the graph onto paper.

i $f(x) = 2x^3 + 5x^2 - 2x - 6$

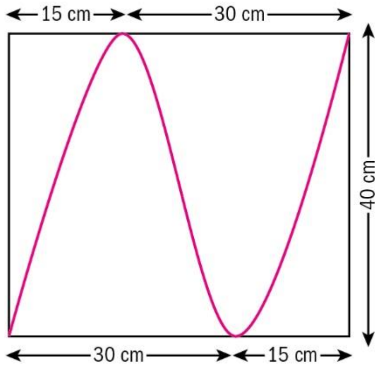
ii $g(x) = (x - 3)^3 + 2$

b Find the coordinates of the x- and y-intercepts.

c Find the coordinates of any local maximum or minimum points.

d Find the inverse where possible.

2 A flag has the following design. The cubic function has rotational symmetry about the point (22.5, 20).



Find the equation of the cubic function.

3 A cardboard pizza box is made from a sheet of cardboard measuring 30 cm by 30 cm with squares of side x cm cut out of the corners.

a Find an expression for the volume, *V*, of the pizza box in terms of x.

b Sketch the graph of your function in **a**.

c Find the x-intercepts and interpret them.

d Find the local maximum and minimum points and the values of x where these occur.

e Explain which value for your answer to **d** is not possible.

f For the possible value of your answer to **d**, find the dimensions of the pizza box and comment on your answer.

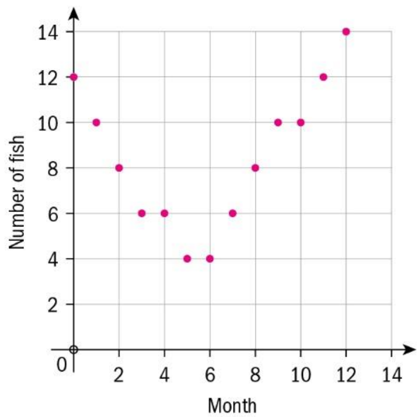
4 Lolo invested 300 euros in shares. The following table shows the average value of Lolo's shares each month for one year.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Amount	300	280	275	270	280	285	305	315	320	325	320	310	295

a Find the best fit cubic equation through these points.

b State whether or not the equation is a good fit for the data and justify your answer.

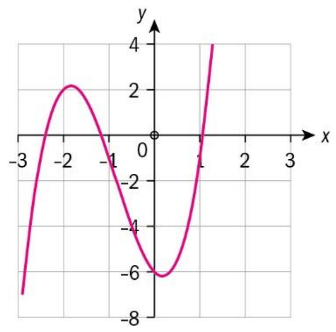
5 Yarkin wanted to find an equation to model the number of fish in the garden pond. He counted the number of fish each week for 12 weeks. The results are shown in the diagram.



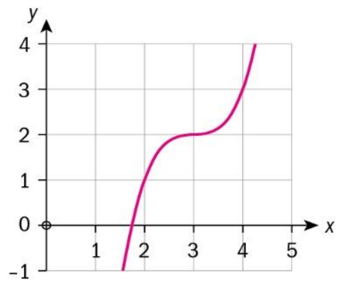
Discuss whether it would be more appropriate for Yarkin to model these data with a quadratic or a cubic model, and justify your choice.

Answers

1 a i



ii



1 b i (-2.39, 0), (-1.17, 0), (1.07, 0) and (0, -6)

ii (1.74, 0)

c i (-1.85, 2.15) and (0.180, -6.19)

ii None.

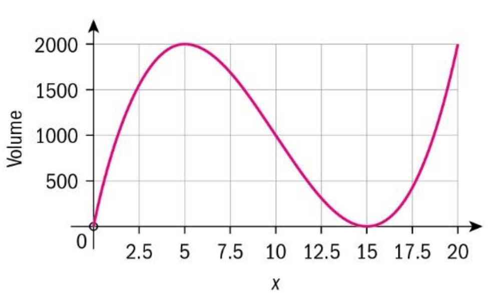
d i no inverse

ii $\sqrt[3]{(x-2)}$ + 3

2 $f(x) = 0.0079x^3 - 0.533x^2 + 8.89x$

3 a $V = x(30 - 2x)^2$

b



c (0, 0) and (15, 0) and x must be greater than 0 and less than 15 for the volume to be realistic.

d (5, 2000) and (15, 0)

e 0 is not a possible answer or you do not have a box.

f The dimensions are 5 by 20 by 20 which is a reasonable size for a pizza box.

4 **a** $f(x) = -0.339x^3 + 5.97x^2 - 23.4x + 299$

b $r^2 = 0.979$ so, the coefficient of determination shows an almost perfect fit, the equation is a very good fit.

5 Quadratic regression gives a coefficient of determination, $r^2 = 0.917$.

Cubic regression gives a coefficient of determination, $r^2 = 0.945$.

So, although both are a good fit, the cubic is the better of the two.

6.4 Power functions, inverse variation and models

1 Sketch the graphs of $f(x) = x^2$ and $g(x) = x^4$ for $-3 \leq x \leq 3$ and compare the two graphs.

2 **a** Sketch the graph of $f(x) = x^4 + 4x^3 - 17x^2 - 24x + 36$.

b Find the coordinates of the points where the graph cuts the axes.

c Find the coordinates of the local maximum and minimum points.

3 The measurements for a pergola are given in the table.

Horizontal distance, cm	0	10	20	30	40	50	60	70	80	90	100
Vertical distance, cm	0	50	100	140	180	180	180	140	100	50	0

a Find the best fit quadratic equation through these points.

b Find the best fit quartic equation through these points.

c Which equation is the best fit for the points? Justify your answer.

4 The distance, *d* km, that a person can walk varies directly as the time, *t* hours.

A person can walk 1 km in 10 minutes.

a Find an equation connecting *d* and *t*.

b Find how many km a person can walk in 1.5 hours.

5 The circumference of a circle, *C*, varies directly as the radius, *r*. When the radius is 2 the circumference is 12.566.

a Find an equation in *C* and *r*.

b Find *C* when the radius is 8.

6 The amount of money, *m*, that each person receives from a lottery win varies inversely with the number of people, *p*, sharing it. If 20 people are sharing the win they each receive $500.

a Find an equation connecting *m* and *p*.

b If there are 30 people sharing the win, find out how much each person receives.

7 Given that *y* varies inversely as the square root of x and that when $x = 16$, $y = 2$.

a Find an equation connecting y and x.

b Find the value of y when $x = 49$.

c Find the value of x when $y = 6$.

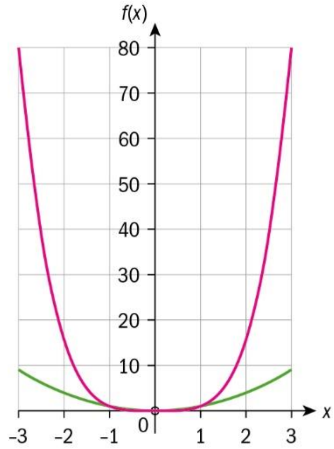
8 Find the inverses of the following functions:

a $f(x) = \frac{1}{2x}$ for $x > 0$

b $f(x) = \frac{4}{x^2}$ for $x > 0$

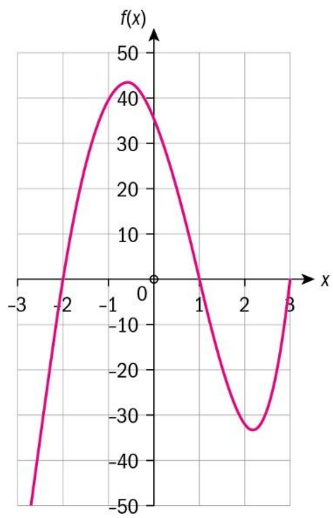
Answers

1



The graph of $f(x) = x^4$ resembles a parabola with a flat base.

2 **a**



b (-2, 0), (1, 0), (3, 0) and (0, 36)

c (-0.603, 43.5) and (2.18, -33.1)

3 **a** $f(x) = -0.0737x^2 + 7.37x - 8.67$.

b $f(x) = 0.0000128x^4 - 0.00256x^3 + 0.0866x^2 + 4.16x + 0.559$.

c For **a**, the coefficient of determination, $r^2 = 0.983$, is an almost perfect fit so this equation is a good fit for the data.

For **b**, the coefficient of determination, $r^2 = 0.997$, is an almost perfect fit so this equation is an even better fit for the data.

4 **a** $d = 6t$

b 9 km

5 **a** $C = 6.283r$

b C = 50.264.

6 **a** $m = \frac{10000}{p}$

b m = 333.33

7 **a** $y = \frac{8}{\sqrt{x}}$

b $y = \frac{8}{7}$

c $x = \frac{16}{9}$

8 **a** $f^{-1}(x) = \frac{1}{2x}$

b $f^{-1}(x) = \pm\frac{2}{\sqrt{x}}$

7.1 Geometric sequences and series

1 The numbers 16, -8, 4, … form the first three terms of a geometric sequence.

a Write down the first term u_1 and the common ratio r.

b Find the 6th term of this sequence.

c Find an expression for the nth term of this sequence.

d Find the sum of the first 12 terms.

2 An ivy plant is 15 cm tall when planted. Every week it grows by 8%, forming a geometric sequence.

a Find how tall it is at the end of the first week.

b Find the value of the common ratio.

c Find the height of the plant at the end of 10 weeks.

3 Kerim buys a bicycle for 450 euros. Each year the value of the bicycle depreciates by 10%.

a Find how much the bicycle is worth at the end of six years.

b Kerim will replace the bicycle when it is worth 150 euros. Find out in how many years Kerim will replace the bicycle.

4 Aron and Asta have 14 days to train for a marathon. Aron trains 45 minutes during the first day and increases the time he trains by 5 minutes every day.

a Calculate how long he trains on day 14.

b Calculate the total length of time he trained over the 14 days.

c Asta also trains 45 minutes the first day. She increases the time that she trains each day by 5%. Calculate how long she trains on day 14.

d Calculate the total length of time she trained over the 14 days.

5 A geometric sequence has second term 1458 and fourth term 162. All the terms are positive.

a Find u_1 and r.

b Find u_6.

c Find u_n.

d Find the sum of the first eight terms.

e Find the sum to infinity.

6 The n^th^ term of a geometric sequence is $u_n = 48(0.45)^n$.

a Write down the first three terms of this sequence.

b Find the sum to infinity of the sequence.

7 Emke buys a painting for $2000. Each year the value of the painting increases by 6%.

Find the value of the painting at the end of five years.

8 Shreya is organising a fun race. The winner is awarded $10, the second is awarded $8, the third is awarded $6.40.

a Show that the amount awarded forms a geometric sequence.

b There is no limit to the number of people taking part in the race. Find out how much Shreya has to pay in total for the prizes.

9 A video is watched by 6 people on the first day, 36 people on the second day, 216 people on the third day and so on.

a Show that these numbers form a geometric sequence.

b Find the number of people who watched the video on the 10^th^ day.

c Can you find the sum to infinity of this sequence? Explain your answer.

10 The sum of an infinite series is 1280. It has a common ratio of 0.75.

Find the first term.

11 a Find $\sum_{r=1}^{7} 24(1.25)^r$

b Find $\sum_{r=0}^{\infty} 240(0.85)^r$

c Find the sum of this geometric series: 2 − 4 + 8 − 16 + … -4096 .

Answers

1 **a** 16, −0.5

b -0.5

c $16(-0.5)^{n-1}$

d 10.66

2 **a** 16.2

b 1.08

c 32.4

3 **a** 239.15

b 10.4 years

4 **a** 110

b 1085 minutes

c 84.9

d 881.9 minutes

5 **a** $r = \frac{1}{3}$, first term = 4374

b 18

c $u_n = 4374(\frac{1}{3})^{n-1}$

d 6560

e 6561

6 **a** $u_1 = 21.6$, $u_2 = 9.72$ and $u_3 = 4.374$

b 39.3

7 2676.45

8 **a** $\frac{8}{10} = 0.8$, $\frac{6.4}{8} = 0.8$ so, r = 0.8

b 50

9 **a** $\frac{36}{6} = 6$, $\frac{216}{36} = 6$, so, r = 6

b 60466176

c No, because r is greater than 1

10 320

11 **a** 91.6

b 1600

c -2730

7.2 Financial applications of geometric sequences and series

1 Chloe deposits \$ 8000 in a bank that pays interest at a rate of 2.3% compounded annually.

Work out how much she has in the bank at the end of 6 years.

2 Stijn invests 1650 euros in a bank that offers interest at a rate of 1.89% compounded annually.

Work out how long it will take for his money to double.

3 Helma invests \$ 5500 in a bank that offers interest at a rate of r% compounded quarterly.

After 5 years Helma has \$ 6300.

a Find r.

b Work out how long it will take for her money to double.

4 Terry invests 12000 CAD in a bank that offers interest at a rate of 1.78% per annum compounded monthly.

a Find out how much he has in the bank at the end of 7 years.

b Find out how long it will take before he has 15 000 CAD in the bank.

5 A bank offers interest at a rate of r% per annum compounded quarterly. Taylor invests

30000 HKD in this bank. After 10 years she has 35 500 HKD in the bank.

a Work out the interest rate.

b At the end of the 10 years Taylor removes her money from the bank and puts it into another bank that offers a rate of 2.61% per annum compounded monthly. Work out how much she has in the bank after another ten years.

6 Jintana invests \$ 2000 in a bank that offers interest at a rate of 2.15% per annum compounded half-yearly. Bente, invests her \$ 2000 in another bank that offers interest at a rate of 2.145% per annum compounded quarterly. Silvana invests her \$ 2000 in yet another bank that offers interest at a rate of 2.14% compounded monthly. Calculate how much each has in the bank after 20 years.

7 The inflation rate of a country is 1.19% per annum.

a If this same rate continues for the next five years, calculate the percentage increase due to inflation at the end of the five years.

b An article costs 50 euros today. Find out how much you would expect it to cost in 2 years' time due to inflation.

8 Colin wants to save for his pension. He saves \$800 every year for 30 years at 2.8% interest compounded annually. Work out how much he will have at the end of the 30 years.

9 Sonyun has received an annuity of 250000 JPY for ten years at 4.6% per annum. The annuity has to be paid out at monthly intervals. Find Sonyun's monthly payments.

10 Hamza borrows \$5000 from a bank that charges 3.9% interest compounded annually. He wants to pay back the loan in 12 monthly instalments.

a Work out how much he has to pay back each month.

b After six monthly instalments, Hamza decides to pay the remaining amount in one payment. Find how much he still owes.

Answers

1 9169.46

2 Just over 37 years

3 **a** 2.73%

b 25.5 years

4 **a** 13591.09 CAD

b 12.5

5 **a** 1.6869%

b 46074.02

6 Jintana has 3067.47

Bente has 3067.92

Silvana has 3067.20

7 **a** 6.09%

b 51.20

8 36850.81

9 2391.01

10 **a** 425.52

b 2524.33

7.3 Exponetial functions and models

1 For each of the following functions:

$f(x) = 3^x - 2$

$f(x) = 4(2)^x + 3$

$f(x) = 3(0.8)^x - 1$

a Write down the coordinates of the point where the curve crosses the y-axis

b The equation of the horizontal asymptote.

c State whether the function is increasing or decreasing.

2 The temperature, T°C, of an apple pie when it comes out of the oven can be modelled by the equation $T(x) = 155(0.75)^x + 20$ where x is the time in minutes.

a Find the initial temperature of the apple pie.

b Find the temperature after 6 minutes.

c Find the time when the temperature is 80°C.

d Write down the equation of the horizontal asymptote.

e Write down the temperature of the room.

3 Patrick invests 5000 euros in a bank that pays interest at a rate of 1.89% per annum compounded quarterly.

a Sketch a graph to model how much money Patrick has in the bank after x years.

b Is an exponential model an appropriate model to use in this case? Explain your answer.

c Use your graph to calculate how long it will take Patrick to have 6000 euros in the bank.

d Check your answer to part c by using the finance app on your GDC.

4 The temperature, T°C, of a cup of hot chocolate is taken every 2 minutes for half an hour

Time	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
Temperature	99	80	65	50	42	36	32	29	27	25	24	23	23	22	22	21.5

a Plot these points on a graph.

b Do you think that an exponential model would be a good fit for these data?

c What do you estimate is the temperature of the room?

d Find the best fit exponential model for this data in the form $T(x) = a(b)^x + c$ where c is your answer to **c**.

5 Find the exact value for *a* if the following piecewise functions are continuous for $x > 0$.

a

$$f(x) = \begin{cases} x^2 + a & \text{for } x < 2 \\ 2^{x+1} & \text{for } x \leq 2 \end{cases}$$

b

$$f(x) = \begin{cases} ax^3 + 10 & \text{for } x < 3 \\ 4^x & \text{for } x \leq 3 \end{cases}$$

Answers

1 a (0, -2) (0, 7) (0, 2)

b $y = -2$ $y = 3$ $y = -1$

c increasing increasing decreasing

2 a 175

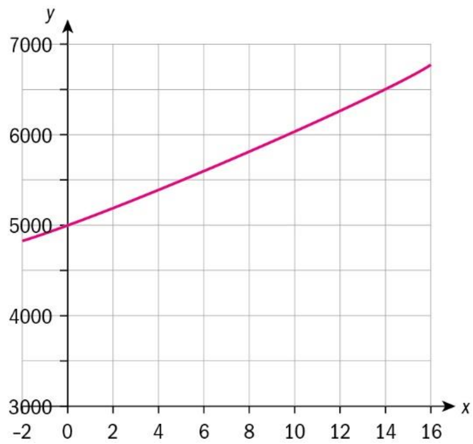
b 47.6

c 3.3

d $y = 20$

e 20°C

3 a

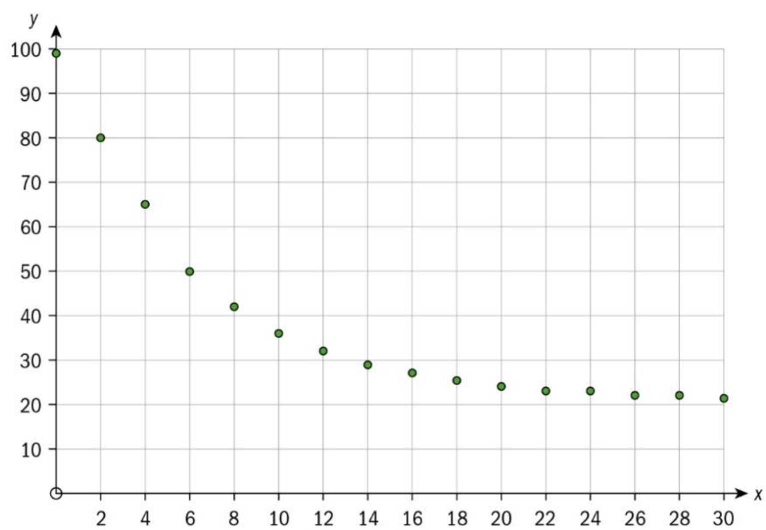


b Yes, because the exponent is the variable.

c 9.67 years

d Checked using GDC app

4 a



b Yes, it looks like an exponential curve.

c About 21°C

d $T(x) = 7.5(0.849)^x + 21$

5 **a** a = 4

b a = 2

7.4 Laws of exponents – laws of logarithms

1 Find the value of:

a log (1000)

b 5 + log (3.2)

c 6.8+ ln (5.3)

d ln (2.1) + log (8.7).

2 Find x if:

a $10^x = 8$

b ln (4 x) = 3.4

c $e^{2x} = 10.1$

d 2log (3 x) = 2.6.

3 Write the following as a single logarithm:

a 4log (x)

b log (x) + 2log (y)

3 + log (x)

d $\frac{3\log(x)}{2}$

e 1 – log (x)

f ln (x) – 3ln (y)

g 3ln (x) – 2.

4 **a** Sketch the graph of the function $f(x) = 10^x - 2$.

b Find its inverse function $f^{-1}(x)$.

c Sketch $f^{-1}(x)$ on the same diagram showing clearly any asymptotes.

5 The following table shows the literacy rate and the GDP for a random selection of countries.

Literacy rate, x%	99.4	99.3	99	99.2	98.7	98.6	98.1	99.1
GDP, \$$y$	19390600	12014610	4872135	3684816	2054969	1379548	637717	396457

Literacy rate, x%	96.3	94.5	86.6	88.5	77.6	66.5	65	69.5	58.4
GDP, \$$y$	313419	102311	47032	22975	13001	4797	844	321	40

a Add another row expressing the GDP as a logarithm.

b Using technology, plot the literacy rate against the logarithm of the GDP.

c Find the equation of the best fit line through your points.

Answers

1 **a** 3

b 5.51

c 8.47

d 1.68

2 **a** 0.903

b 7.49

c 1.16

d 6.65

3 **a** $\log\,x^4$

b $\log\,(xy^2)$

c $\log\,(1000x)$

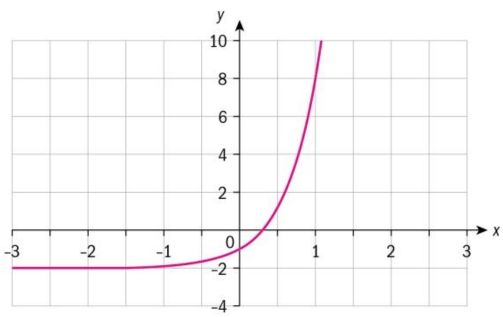
d $\log\,\frac{x^3}{100}$

e $\log\,\frac{10}{x}$

f $\ln\frac{x}{y^3}$

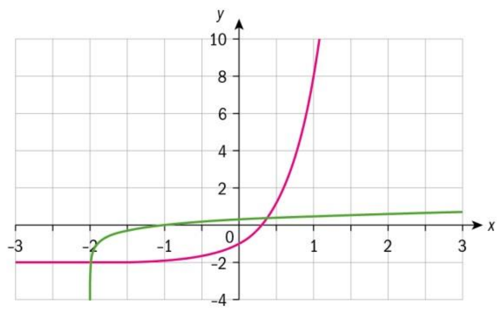
g in $\frac{x^3}{e^2}$

4 **a**



b $y = \log(x + 2)$

c

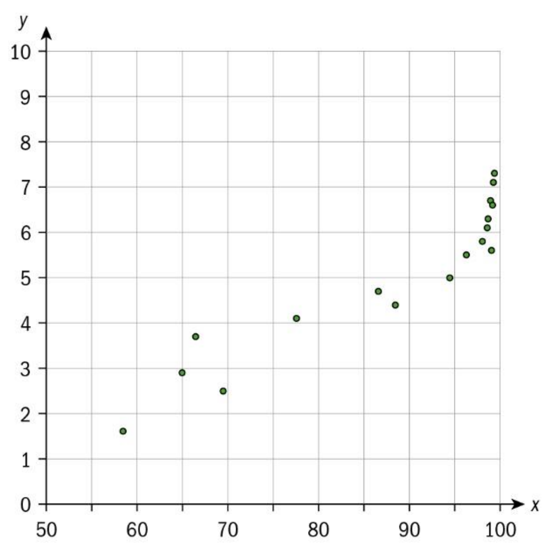


5 a

Literacy rate, x%	99.4	99.3	99	99.2	98.7	98.6	98.1	99.1
GDP, y\$	19390600	12014610	4872135	3684816	2054969	1379548	637717	396457
log (y)	7.288	7.080	6.688	6.566	6.313	6.140	5.805	5.598

Literacy rate, x%	96.3	94.5	86.6	88.5	77.6	66.5	65	69.5	58.4
GDP, y\$	313419	102311	47032	22975	13001	4797	844	321	40
log (y)	5.496	5.010	4.672	4.361	4.114	3.681	2.926	2.507	1.602

b



c Linear model is $f\left(x\right) = 1.07x - 4.37$ with $r^2 = 0.880$

Quadratic model has $r^2 = 0.889$

Cubic model has $r^2 = 0.924$

Quartic model has $r^2 = 0.929$

They are all quite a reasonable fit with the cubic and quartic models having the greatest coefficient of determination.

7.5 Logistic models

1 Evaluate $f\left(x\right)=\dfrac{120}{1+8e^{-2x}}$ for $x = -6,\ -2,\ 0,\ 2, 6$.

2 From 2010 until 2018, the number of people over 65 years old in the Netherlands who own a smartphone can be modelled by the equation $f\left(x\right)=\dfrac{3200000}{1+8e^{-0.32x}}$. .

Use the model to estimate how many people over 65 owned a smartphone in 2010.

b Use the model to estimate how many people over 65 owned a smartphone in 2018.

c It is predicted that no more than 90% of over 65-year-old people will ever own a smartphone; use the model to find out when this will occur.

3 The length of a pumpkin plant is measured, in metres, every week for 12 weeks.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Length	1	1.3	1.6	2.4	2.9	3.5	4.8	5.6	6.1	6.5	6.8	7

a Plot these data points on graph paper.

b Using technology, find a best-fit logistic model for these data.

c Sketch your equation from **b** on the same diagram as **a** and comment on the suitability of the model.

Answers

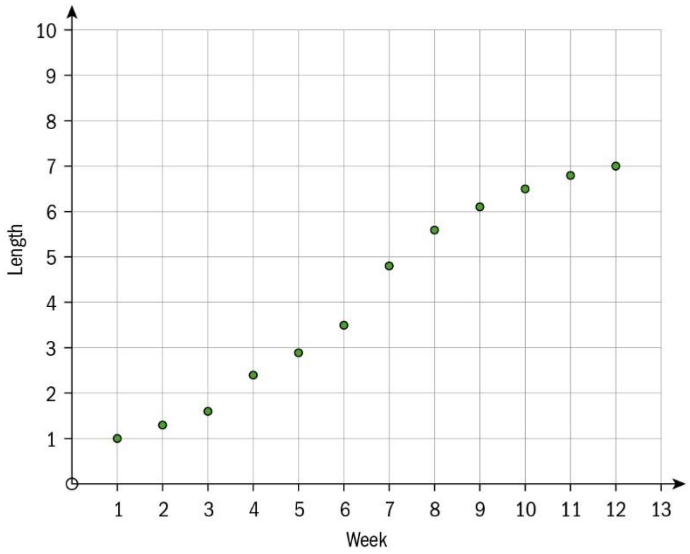
1 0.0000922, 0.274, 13.3, 105, 120

2 **a** 355556

b 1977215

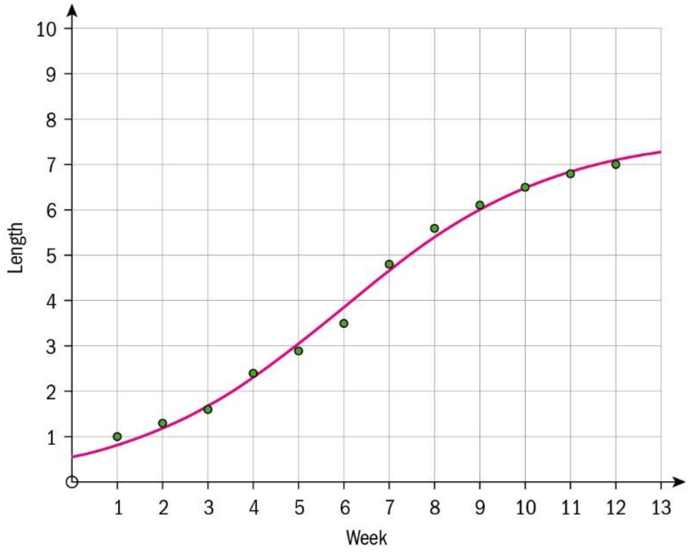
c After 13.4 years, so in 2024

3 **a**



b $y = \dfrac{7.627}{1 + 12.86e^{-0.429x}}$

c



Yes, it is a very good fit for the data.

8.1 Measuring angles

1 Inflatable Zorb Hamster Balls are made in two sections, one sphere inside another with layers of air in between. The two spheres are linked with a network of wires. Find:

a the angle between the wires straps around the circumference

i in degrees

ii in radians

b the area of the sectors created if the diameter of the exterior sphere is 3 m

c the area of air between the two spheres if the radius of the interior sphere is 1 m.

Number of attaches	2	3	4	6	7	9	10	15	17	19	n
Angle in degree											
Angle in radian											
Area of sectors											
Area between the 2 spheres											

2 **a** Find the distance travelled if the ball has rolled:

i 1^c **ii** 3^c **iii** 4.2^c

iv 5.9^c **v** 6.3^c **vi** 100^c.

b For each of the above cases, state:

i when the ball rolled more than once

ii how many turns and extra degrees it rolled.

3 The radii of the tunnel entrances on the exterior sphere are 30 cm on one side and 45 cm on the other. Find:

a the angle of the exterior sphere it represents in radian and in degrees

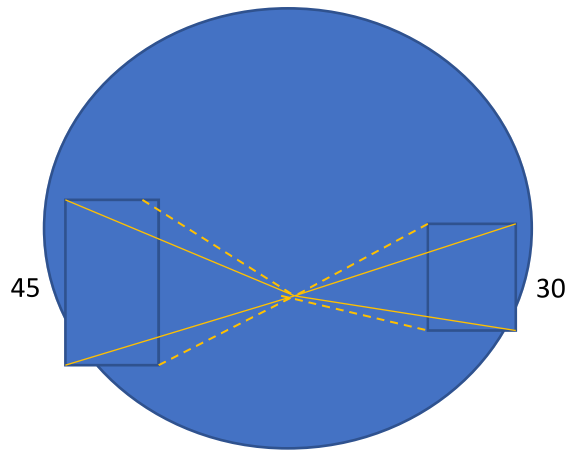
b the angle of the interior sphere it represents in radian and in degrees

c the length of the missing arcs on the exterior sphere, write your answer in cm to 2 dp

d the length of the missing arcs on the exterior sphere, write your answer in cm to 2 dp

e the area of the missing circular segment in the exterior sphere, round it to the nearest cm^2

f the area of the missing circular segment in the exterior sphere, round it to the nearest cm^2.



Answers

1

Number of attaches	2	3	4	6	7	9	10	15	17	19	n
Angle in degree	$180°$	$120°$	$90°$	$60°$	$60°$	54.3° (1dp)	$36°$	$24°$	21.2° (1dp)	18.9° (1dp)	$\frac{360°}{n}$
Angle in radian	π	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{7}$	$\frac{2\pi}{9}$	$\frac{\pi}{5}$	$\frac{2\pi}{15}$	$\frac{2\pi}{17}$	$\frac{2\pi}{19}$	$\frac{2\pi}{n}$
Area of sectors	3.53 m^2	2.36 m^2	1.77 m^2	1.18 m^2	1.01 m^2	0.79 m^2	0.7 m^2	0.47 m^2	0.42 m^2	0.37 m^2	$1.125\pi n$ m^2
Area between the 2 spheres	1.96 m^2	1.3 m^2	0.98 m^2	0.65 m^2	0.56 m^2	0.44 m^2	0.39 m^2	0.26 m^2	0.23 m^2	0.21 m^2	0.625 πn m^2

2 **a** **i** 1.5 m **ii** 4.5 m **iii** 6.3 m

iv 8.85 m **v** 9.45 m **vi** 150 m

b **i** to **iv** was less than a full turn

v it did a full turn plus 0.963 degrees

vi it did 15 full turns plus 0.915 degrees

3 Using A = $\arccos\left(\frac{(2b^2 - a^2)}{2b^2}\right)$ or half of the radius and sin ratio

a **i** 0.2 radian or 11.48° **ii** 0.301 radian or 17.25°

b **i** 0.3 radian or 17.25° and **ii** 0.45 radian or 26.01°

c **i** 0.3005 **ii** 30.12

d **i** 45.17 **ii** 45.39

e **i** 20 **ii** 23

f **i.** 51 **ii** 77

8.2 Sinusoidal models: f(x) = asin(b(x-c))+d

1 One of the main performances in a local amusement part is 'Johnny and Sammy the Dare Devil Bikers', a show featuring two bikers inside a hollow sphere 5 m above the ground.

a They enter the sphere from the bottom, driving through a ramp. Their movement on the ramp can be modelled by $y = 0.1(x-3)^3 + 3.1$ where y corresponds to the height of the centre in metres of the front wheel and x the horizontal distance from the bottom of the ramp $x \geq 0$. Use technology

i to graph the shape of the ramp

ii to find the radius of the front wheel in cm

iii to find the horizontal distance between the bottom of the ramp and the centre of the sphere.

b For their first show, they drive up and down on the same planes. Given that the radius of the ball is 10.4 m their movements will be represented on a circle with $r = 10\,m$. J represents Johnny's position, S is Sammy's and the angles of their respective positions measured anticlockwise from the x-axis are θ and α .

i $\theta = \frac{7}{18}\pi$ **ii** $\theta = \frac{5}{36}\pi$ **iii** $\theta = -\frac{7}{36}\pi$ **iv** $\theta = -\frac{\pi}{3}$

v $\theta = \frac{2}{3}\pi$ **vi** $\theta = \frac{7}{6}\pi$ **vii** $\theta = \frac{25}{18}\pi$ **viii** $\theta = 0$

A: For **i** to **viii**, calculate Johnny's position:

a) relative to the centre

b) to the bottom of the ramp

B: If Sammy is at the same height as Johnny

a) Find α for **i** to **viii**

b) Complete: if they are both at the same height $\sin\alpha = \sin(\theta)$ if $\alpha =$

C: If Sammy is right above or below Johnny

a) Find α for **i** to **viii**

b) Complete: if Sammy is right above or below Johnny, $\cos\alpha = \cos(\theta)$ if $\alpha =$

D: If Sammy and Johnny are on diametrically opposite positions

a) Find α for **i** to **viii**

b) Complete: if Sammy and Johnny are on diametrically opposite positions

$\cos\alpha = -\cos(\theta)$ and $\sin\alpha = -\sin(\theta)$ if $\alpha =$

2 On the Gusano loco carousel, children go up and down at a speed and height determined by the stallholder, depending on the age of the people on it.

a The following graphs represent the height in metres against the time in seconds of various rides. For each of them find:

i the height of the start of the ride

ii the range

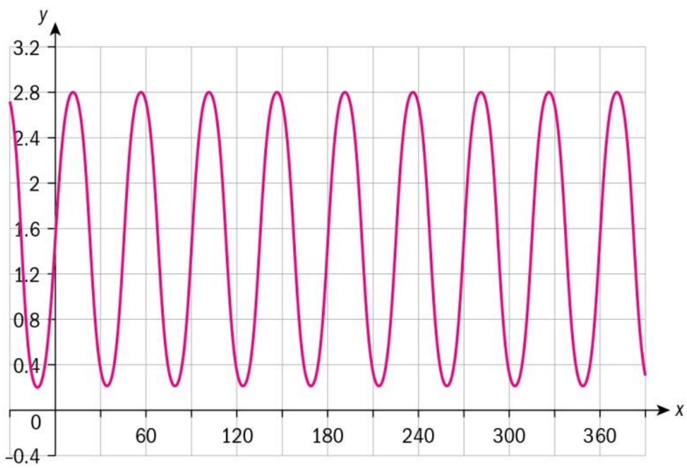
iii the amplitude

iv the equation of the principal axis

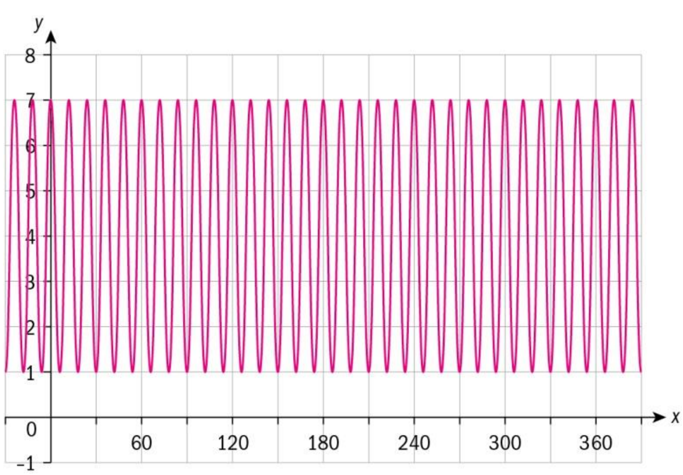
v the period

vi its equation

A

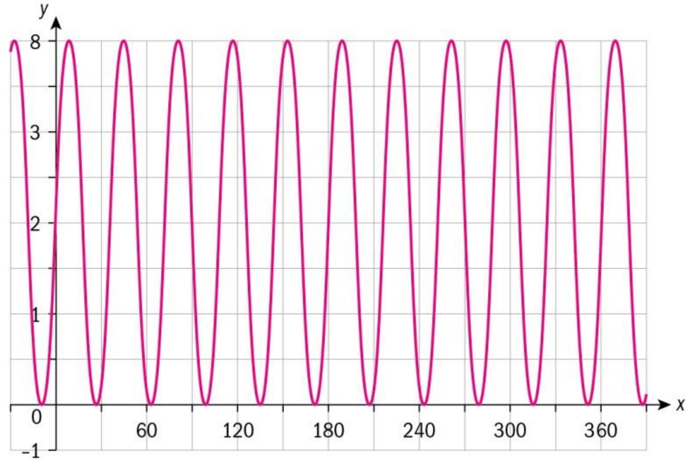


B

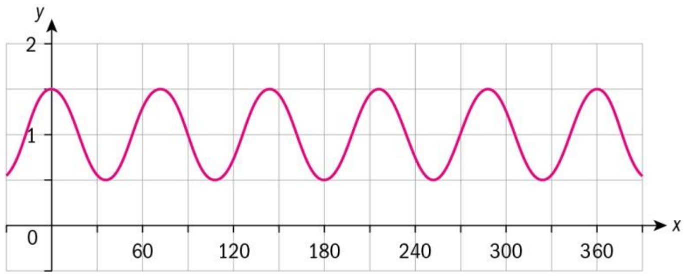


C

C



D



Order the graphs according to their corresponding age groups (with the youngest first).

3 Moscow, Copenhagen and Glasgow are on the same latitude.

The table shows the average rainfall in mm and the months as numbers e.g. Jan = 1

Months	1	2	3	4	5	6	7	8	9	10	11	12	Year
Moscow	34.4	29.0	32.7	38.2	51.0	65.6	81.5	76.8	60.7	50.4	44.1	42.4	598.5.6
Copenhagen	42.3	25.2	26.2	31.0	41.9	52.4	67.0	74.5	71.4	62.8	54.3	51.3	603.1
Glasgow	91.1	78.2	71.0	60.2	65.7	73.0	94.7	108.0	113.7	115.7	116.2	106.7	1094.2

Plot the data.

a For each graph find:

i the period

ii the range

iii the amplitude

iv the equation of the principal axis

v the average quantity of rainfall.

b Hence write the equations matching each graph.

c Discuss why there is at times a discrepancy between the value of the principal axis and the average quantity of rainfall.

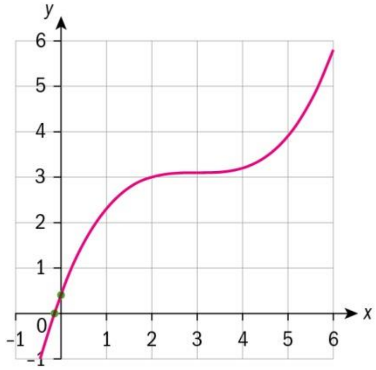
d Use the sine regression of your GDC to check your model.

e State the country that has the most constant amount of rainfall over the year.

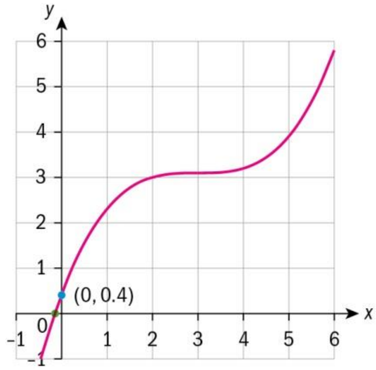
f State when it is the best time to travel to Glasgow, if you wanted less than 70 ml of rain.

Answers

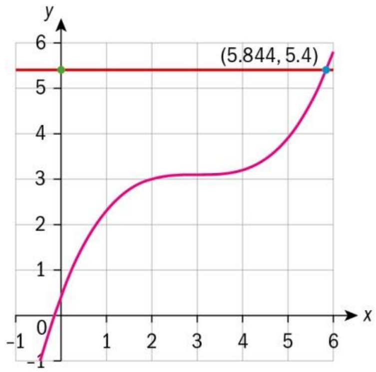
1 a i



ii 0.4 m or 40 cm



iii 5.844 m



b Aa) using $(10\,cos\theta, 10\,\sin\theta)$; all the answers are in m

i $(3.42, 9.4)$ **ii** $(9.06, 4.23)$ **iii** $(8.19\,, -5.73\,)$ **iv** $(5, -8.66\,)$

v $(-5\,, 8.66)$ **vi** $(-8.66\,, -5\,)$ **vii** $(3.42,\ 9.4)$ **viii** $(\,10, 0)$

Ab) using $(10\,cos\theta + 5.844\,,\ 10\sin(\,\theta) + 5)$ all the answers are in m

i $(9.26, 14.3)$ **ii** $(14.9, 9.23\,)$ **iii** $(14.04, -0.74)$ **iv** $(10.844\,, -3.66)$

v $(0.844\,,13.66)$ **vi** $(-2.82, 0\,)$ **vii** $(9.26, 14.3)$ **viii** $(15.844, 5)$

Ba)

i $\frac{11}{18}\pi$ **ii** $\frac{31}{36}\pi$ **iii** $\frac{43}{36}\pi$ **iv** $\frac{4}{3}\pi$

v $\frac{1}{3}\pi$ **vi** $-\frac{\pi}{6}$ **vii** $\frac{7}{18}\pi$ **viii** π

Bb) $\sin\alpha\ = \sin(\theta)$ if $\alpha = (\pi - \theta)$

Ca)

i $-\frac{7}{18}\pi$ **ii** $-\frac{5}{36}\pi$ **iii** $\frac{7}{36}\pi$ **iv** $\frac{1}{3}\pi$

v $-\frac{2}{3}\pi$ **vi** $-\frac{7}{6}\pi$ **vii** $-\frac{25}{18}\pi$ **viii** 0

Cb) $\cos\alpha\ = \cos(\theta)$ if $\alpha = -\theta$

Da)

i $-\frac{11}{18}\pi$ **ii** $-\frac{31}{36}\pi$ **iii** $-\frac{43}{36}\pi$ **iv** $-\frac{4}{3}\pi$

v $-\frac{\pi}{3}$ **vi** $\frac{\pi}{6}$ **vii** $-\frac{7}{18}\pi$ **viii** π

Db) $\sin\alpha\ = \sin(\theta)$ if $\alpha = (\pi + \theta)$

2 a

A) **i** 1.5 **ii** $y \in [0.2, 2.8]$ **iii** 1.3

iv $y = 1.5$ **v** 45 **vi** $y = 1.3\sin 8x + 1.5$

B) **i** 7 **ii** $y \in [1, 7]$ **iii** 3

iv $y = 4$ **v** 12 **vi** $y = 3\cos 30x + 4$

C) **i** 2 **ii** $y \in [0, 4]$ **iii** 2

iv $y = 2$ **v** 36 **vi** $y = 2\sin 10x + 2$

D) **i** 1.5 **ii** $y \in [0.5, 1.5]$ **iii** 0.5

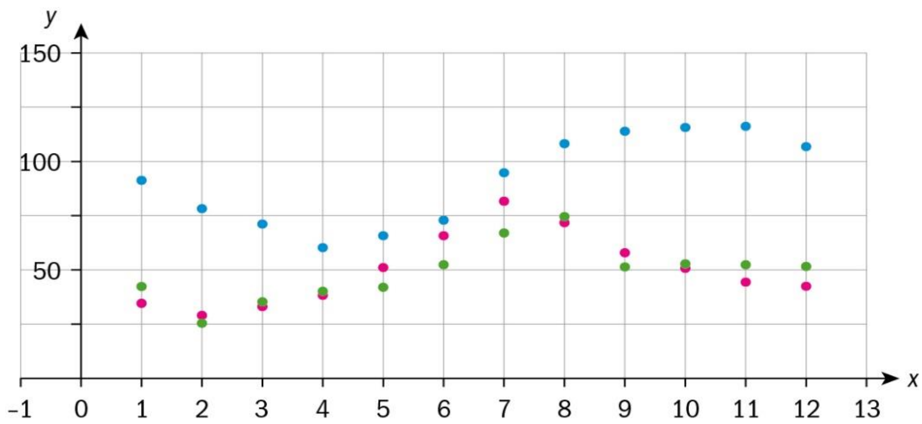
iv $y = 1$ **v** 72 **vi** $y = 0.5\cos 5x + 1$

b D, A, C, B

3 a

x_2	y_2	z_2	u_2
1	34.4	42.3	91.1
2	29.0	25.2	78.2
3	32.7	35.2	71.0
4	38.2	40.0	60.2
5	51.0	41.9	65.7
6	65.6	52.4	73.0
7	81.5	67.0	94.7
8	71.8	74.4	108.0
9	57.7	51.4	113.7
10	50.4	52.8	115.7
11	44.1	52.3	116.2
12	42.4	51.3	106.7

b



c

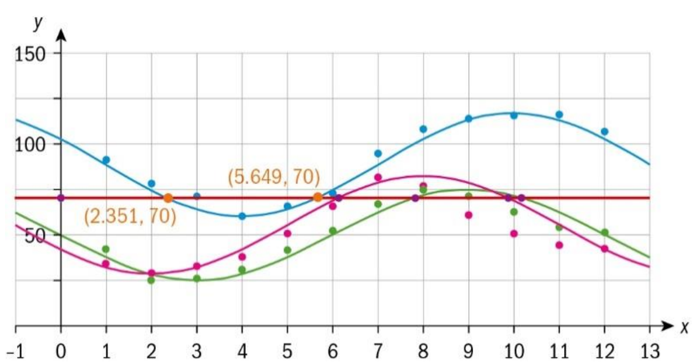
	period	range	amplitude	equation of the principal axis	average quantity of rainfall	equation in the form $y = a\sin bx + d$
Moscow	-30	$[29, 81.5]$	26.25	y=55.25	50.54	$y = 55.25 - 26.25\sin(30(x+1))$
Copenhagen	-30	$[25, 74.5]$	24.65	y=49.85	50	. $y = 49.85 - 24.65\sin(30x)$
Glasgow	-30	$[60.2, 116.25]$	28	y=88.2	81.2	$y = 88.2 - 28\sin(30(x-1))$

d

The difference arises from the two different ways of doing averages.

e See your GDC.

f Copenhagen: smallest amplitude and mean are almost equal to the equation of the principal axis. So, between the 10th February and 20th May.



8.3 Completing our number system

1 State:

a whether the values **i** to **ix** can be placed on the number line

b (all) the set(s) they belong to.

i 4 **ii** 0 **iii** 4.7

iv $\frac{3}{4}$ **v** -2 **vi** 2.4

vii π **viii** $5+6i$ **ix** $7i$

2 State the set of numbers that can represent:

a linear motion

b non-linearity.

3 Calculate:

a $\sqrt{-4}$ **b** $\sqrt{-81}$ **c** $\sqrt{-\frac{1}{49}}$ **d** $\sqrt{-\frac{25}{36}}$

e $\sqrt{-11}$ **f** i^1 **g** i^3 **h** i^4

i i^5 **j** i^{21} **k** i^{4n+1} for $n \in \mathbb{N}$ **l** i^{4n+2} for $n \in \mathbb{N}$

m i^{4n+3} for $n \in \mathbb{N}$ **n** i^{4n+4} for $n \in \mathbb{N}$.

4 Find real numbers a and b such that:

a $a + ib = 8$ **b** $a + ib = 8i$ **c** $a + ib = -49i$

d $a + ib = (8 + 3i) + (3 - i)$ **e** $a + ib = (8 + 3i) - (3 - i)$

f $a + ib = (-8 + 3i) - (3i + 3)$ **g** $a + ib = (5 + 3i)(6i)$

h $a + ib = (2 + 5i)(6 + i)$ **i** $14 + ib = (a - i)(3 + 2i)$

j $a + ib = (7 + 9i)^2$ **k** $a + ib = (9 - 7i)^2$ **l** $a + ib = i(-9 + 7i)^2$

m $a + ib = \frac{1}{2-i}$ **n** $a + ib = \frac{2+8i}{1-i}$ **o** $a + ib = \left(\frac{2+8i}{1-i} + 2 - i\right)^2$.

5 Solve:

a $z^2 + 121 = 0$ **b** $z^2 + 5 = 0$ **c** $z^2 - 9 = 0$ **d** $2z^2 + 47 = 0$

e $z^4 = 16$ **f** $z^2 - 6z + 11 = 0$ **g** $6z^2 + 12z + 5 = 0$ **h** $2\,z^2 + 2z = -7$

i $3z^2 + iz + 4 = 0$ **j** $z^4 = -4z^2$ **k** $z^4 + 10z^2 + 25 = 0$ **l** $z^4 = -6z^2 - 9$.

Answers

1 **a** **i** yes **ii** yes **iii** yes

iv yes **v** yes **vi** yes

vii yes **viii** no **ix** no

b **i** $4 \in \mathbb{N} \in \mathbb{R} \in \mathbb{C}$ **ii** $0 \in \mathbb{R} \in \mathbb{C}$ **iii** 4.7$\in \mathbb{R}^{+} \in \mathbb{C}$

iv $\frac{3}{4} \in \mathbb{Q} \in \mathbb{R} \in \mathbb{C}$ **v** -2$\in \mathbb{N}^{-} \in \mathbb{R} \in \mathbb{C}$ **vi** $4 \in \mathbb{R}^{-} \in \mathbb{C}$

vii $\pi \in \mathbb{R} \in \mathbb{C}$ **viii** 5+6$i \in \mathbb{C}$ **ix** 7$i \in \mathbb{C}$

2 **a** real numbers

b complex numbers, as real numbers only represent one dimension and you need to add the complex part to get a precise description of the motion of the object in both dimensions.

3 **a** $2i$ **b** $9i$ **c** $\frac{i}{7}$ **d** $\frac{5}{6}i$

e $i\sqrt{11}$ **f** i **g** $-i$ **h** 1

i i **j** i **k** i **l** -1

m $-i$ **n** 1

4 **a** $a = 8, b = 0$ **b** $a = 0, b = 8$ **c** $a - 0, b = -49$

d $a = 11, b = 2$ **e** $a = 5,\ b = 4$ **f** $a = -11,\ b = 0$

g $a = -18,\ b = 30$ **h** $a = 7,\ b = 32$ **i** $a = 4,\ b = 5$

j $a = -32, b = 126$ **k** $a = 32, b = -126$ **l** $a = 126, b = 32$

m $a = 0.4,\ b = 0.2$ **n** $a = 5,\ b = 3$ **o** $a = -15, b = -8$

5 **a** $\pm 11i$ **b** $\pm i\sqrt{5}$ **c** ± 3 **d** $\pm \frac{7}{\sqrt{2}}$

e ± 2 or $\pm 2i$ **f** $i,\ i + 1$ **g** $-1 \pm \frac{i\sqrt{21}}{6}$ **h** $-\frac{1}{2} \pm \frac{i\sqrt{13}}{2}$

i $\frac{1}{6} \pm 7i$ **j** 0 or $\pm 2i$ **k** $\pm i5$ **l** $\pm i\sqrt{3}$

8.4 A geometrical interpretation of complex numbers

Navigation systems such as maritime radar use complex numbers in polar form to locate positions in their surroundings

Let $\mathrm{z_n}$ represent the coordinates of n vessels around your boat, with all the measurement in km.

$z_1 = 5$ $z_2 = 3i$ $z_3 = -2$ $z_4 = -1.1i$ $z_5 = 0$

$z_6 = 2\sqrt{3} + 2i$ $z_7 = 2\sqrt{3}i + 2$ $z_8 = \sqrt{3}i - 1$ $z_9 = 1 - \sqrt{3}i$

$z_{10} = -1 - \sqrt{3}i$ $z_{11} = -\frac{1}{4} - \frac{1}{4}i,$ $z_{12} = -\frac{1}{4} + \frac{1}{4}i$ $z_{13} = \frac{1}{\sqrt{2}}(i - 1)$

$z_{14} = \frac{1}{2}(-1 - \sqrt{3}i)$ $z_{15} = \frac{2}{1+i}$ $z_{16} = \frac{2-2i}{1+i}$

1 Plot all the z_n on an argand diagram and hence state their polar forms.

2 State which vessels are:

a within 1.5 km of the boat

b exactly 2 km away

c collinear

d images of each other in a reflection about:

i the $\mathrm{Im}(z)$axis **ii** the $\mathrm{Re}(z)$ axis

e images of each other in a rotation of

i $\frac{\pi}{2}$ about the origin **ii** $\frac{\pi}{3}$ about the origin **iii** $\frac{5\pi}{12}$ about the origin.

3 Complete the sentence. If two boats $z_\alpha = a_\alpha + ib_\alpha = \lambda_\alpha \mathrm{cis}(\theta_\alpha)$ and $z_\beta = a_\beta + ib_\beta = \lambda_\beta \mathrm{cis}(\theta_\beta)$ are:

a the image of each other in a reflection in the y axis then $a_\alpha + ib_\alpha = \cdots$ and $\lambda_\alpha \mathrm{cis}(\theta_\alpha) = \cdots$

b the image of each other in a reflection in the y axis then $a_\alpha + ib_\alpha =$ and $\lambda_\alpha \mathrm{cis}(\theta_\alpha) =$

c collinear $a_\alpha - ib_\alpha = \cdots$ for all all $k \in \mathbb{R}$ and $\lambda_\alpha \mathrm{cis}(\theta_\alpha) =$

4 For the boat $z_\alpha = a_\alpha + ib_\alpha = \lambda_\alpha \mathrm{cis}(\theta_\alpha)$, explain the geometrical transformation created by multiplying its complex and polar notation by

a i

b i^2

c i^3.

5 **a** Explain the geometrical interpretation of $\lambda_\alpha \text{cis}(\theta_\alpha) \times \text{cis}\left(\frac{\pi}{2}\right)^n$ with $n \in \mathbb{N}$.

b Write the result in polar form.

c Explain the geometrical interpretation of the conjugate of z_α.

6 Find:

a $\text{cis}\left(\frac{\pi}{3}\right)$ in the form $a + ib$

b **i** $z_3 \times \text{cis}\left(\frac{\pi}{3}\right)$

ii $z_{10} \times \text{cis}\left(\frac{\pi}{3}\right)$ write your answer in Cartesian and polar form

c $\text{cis}\left(\frac{k\pi}{3}\right) \times r\text{cis}\left(\frac{q\pi}{3}\right) = \cdots$. with $k, q \in \mathbb{R}$

d w such that zw is the coordinate of a boat with original position z after a rotation of $\frac{7\pi}{9}$.

7 A boat in initial position $z = \frac{1}{2}e^{\left(\frac{5\pi}{6}\right)}$ will go through $\left(\frac{1}{2}e^{\left(\frac{5\pi}{6}\right)}\right)^n$ after n minutes

a Write $\frac{1}{2}e^{\left(\frac{5\pi}{6}\right)}$ in:

i $r\text{cis}(d\pi)$ form

ii Cartesian form.

b Calculate z^n for:

i $n = 1$

ii $n = 2$

iii $n = 3$.

Write your answer as $a + ib$,$\frac{1}{c}\text{cis}\left(\frac{5d\pi}{6}\right)$ and $me^{\left(\frac{5p\pi}{6}\right)}$.

c Plot your result.

d Suggest a description for the path of the boat.

e Find when the boat is:

i due West

ii due South.

8 Write the position of a boat z as a complex number if:

a $z = \text{e}^{-i\pi}$

b $z = \text{e}^{-i}$

c i^{-i}.

9 Let $z_n = \overrightarrow{OZ_n}$ be the movement of boat n without current and $c_n = \overrightarrow{OC_n}$ with the current.

a Explain why $z_n + c_n$ gives the actual movement of the boat.

b Find the actual speed and angle of the boat if:

i $z_1 = 2 - i$ and $c_1 = z_1^*$

ii $z_1 = 4(\sqrt{2} - \sqrt{6}i)$ and $c_1 = 4(-\sqrt{6} + \sqrt{2}i)$.

c Estimate $|z_n|$ and Arg(z_n) in degrees if:

i the actual movement is 6i and $c_1 = 1$

ii the actual movement is $3 + 6$i and $c_1 = -1 + 2i$.

Answers

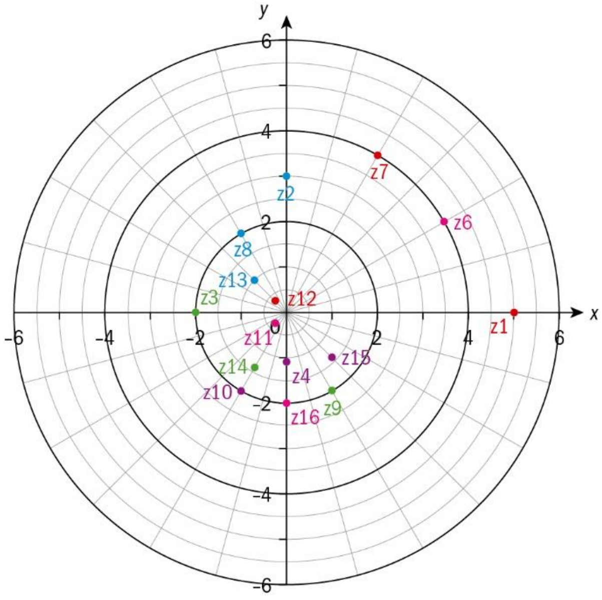
1 $z_1 = 5\text{cis}\,(0)$, $z_2 = 3\text{cis}\left(\frac{\pi}{2}\right)$ $z_3 = -2\ \text{cis}(\pi)$ $z_4 = 1.1\text{cis}\left(-\frac{\pi}{2}\right)$

z_5 =*zero has no polar form* $z_6 = 4\text{cis}\left(\frac{\pi}{6}\right)$ $z_7 = 4\text{cis}\left(\frac{\pi}{3}\right)$ $z_8 = 2\text{cis}\left(\frac{2\pi}{3}\right)$

$z_9 = 2\text{cis}\left(\frac{5\pi}{3}\right)$ $z_{10} = 2\text{cis}\left(\frac{4\pi}{3}\right)$ $z_{11} = \frac{\sqrt{2}}{4}\text{cis}\left(\frac{5\pi}{4}\right)$, $z_{12} = \frac{\sqrt{2}}{4}\text{cis}\left(\frac{3\pi}{4}\right)$

$z_{13} = 1\text{cis}\left(-\frac{\pi}{4}\right)$ $z_{14} = 1\text{cis}\left(-\frac{2\pi}{3}\right)$ $z_{15} = \frac{2}{1+i}.\frac{1-i}{1-i} = (1-i) = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$ $z_{16} = -2i = 2\text{cis}\left(-\frac{\pi}{2}\right)$

https://www.desmos.com/calculator/ovk6whkjqv



2 **a** $z_4, z_{12}, z_{11}, z_{12}, z_{13}, z_{15}$ **b** $z_3, z_8, z_9, z_{10}, z_{16}$

c z_2, z_4 *and* z_{16} are collinear z_7, z_{10}, *and* z_{14} are collinear, z_1, *and* z_3 are collinear,

z_{12}, z_{13} *and* z_{15} are collinear, z_9, ***and*** z_8 are collinear

d **i** z_9, z_{10} **ii** z_8, z_{10}

e **i** z_3, z_{16} **ii** z_8, z_3, z_{10}, z_9 **iii** z_{14}, z_{15}

3 **a** $a_\alpha + ib_\alpha = -a_\beta + ib_\beta$ or $\lambda_\alpha\text{cis}(\theta_\alpha) = \lambda_\beta\text{cis}(\pi - \theta_\beta)$

b $a_\alpha + ib_\alpha = a_\beta - ib_\beta$ or $\lambda_\alpha\text{cis}(\theta_\alpha) = \lambda_\beta\text{cis}(-\theta_\beta)$

c $a_\alpha + ib_\alpha = k(a_\beta + ib_\beta) = kz_\beta$ for all all $k \in \mathbb{R}$ and $\lambda_\alpha\text{cis}(\theta_\alpha) = \lambda_\beta\text{cis}(\pi + \theta_\beta)$

4 **a** $z_\alpha = ia_\alpha - b_\alpha$ rotation $\frac{\pi}{2}$ **b** $z_\alpha = -a_\alpha - ib_\alpha$ rotation π **c** $z_\alpha = -ia_\alpha + b_\alpha$ rotation $\frac{3\pi}{2}$

5 **a** if $n = 4k+1$, $\lambda_\alpha\text{cis}(\theta_\alpha) \times \text{cis}\left(\frac{\pi}{2}\right)^n = z_\alpha \times i$ so it is a rotation $\frac{\pi}{2}$

b if $n = 4k+2$, $\lambda_\alpha\text{cis}(\theta_\alpha) \times \text{cis}\left(\frac{\pi}{2}\right)^n = z_\alpha \times -1$ rotation π,

c if $n = 4k + 3$, $\lambda_\alpha \text{cis}(\theta_\alpha) \times \text{cis}\left(\frac{\pi}{2}\right)^n = z_\alpha \times -i$ rotation $\frac{3\pi}{2}$

6 a $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

b i $-2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i = 2\text{cis}\left(\frac{4\pi}{3}\right) = z_{10}$

ii $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-1 - \sqrt{3}i\right) = 1 - \sqrt{3}i = 2\text{cis}\left(\frac{5\pi}{3}\right) = z_9$

c $r\text{cis}\left(\frac{(q+k)\pi}{3}\right)$

d $r = 1 \text{ } and \text{ } \theta_w = \frac{7\pi}{9}$

7 a i $\frac{1}{2}\text{cis}\left(\frac{5\pi}{6}\right)$ **ii** $-\frac{\sqrt{3}}{4} + \frac{i}{4}$

b i $\frac{1}{8} - \frac{\sqrt{3}}{8}i = \frac{1}{4}\text{ cis}\left(\frac{2\times5\pi}{6}\right) = \frac{1}{4}e^{\left(\frac{2\times5\pi}{6}\right)}$

ii $\frac{i}{16} = \frac{1}{8}\text{cis}\left(\frac{3\times5\pi}{6}\right) = \frac{1}{8}e^{\left(\frac{3\times5\pi}{6}\right)}$

iii $\frac{-1}{32} - \frac{\sqrt{3}}{32}i = \frac{1}{16}\text{cis}\left(\frac{4\times5\pi}{6}\right) = \frac{1}{16}e^{\left(\frac{4\times5\pi}{6}\right)}$

c https://www.desmos.com/calculator/d38jir0liu

d spiralling in

e i if it is due West $\left(\frac{1}{2}cis\left(\frac{5\pi}{6}\right)\right)^n = \left(\frac{1}{2}(\cos(\pi) + i\sin(\pi))\right)^n$, $\frac{5n\pi}{6} = \pi$ if $n = \frac{6}{5}$ minutes

ii if it is due South $\left(\frac{1}{2}cis\left(\frac{5\pi}{6}\right)\right)^n = \left(\frac{1}{2}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)\right)^n$, $\frac{5n\pi}{6} = \frac{3\pi}{2}$ if $n = \frac{9}{5}$ minutes

8 a -1, **b** $e^{-i\times1} = \text{cis}(1) = 0.540 - 0.841i$ **c** $\left(e^{\left(i\frac{\pi}{2}\right)}\right)^{-i} = e^{\left(\frac{\pi}{2}\right)} = 4.81$

9 a $4\text{cis}(0)$ speed =4 angle=0

b $4\text{cis}\left(\frac{7\pi}{12}\right)$ speed =4, angle $=\frac{7\pi}{12}$

c i $|z_n| = \sqrt{37}$ and $\text{Arg}(z_n) = 80.54°$ **ii** $|z_n| = \sqrt{68}$ and $\text{Arg}(z_n) = 75.96°$

8.5 Using complex numbers to understand periodic models

Biorhythms are believed to be cyclic patterns of physical, emotional and intellectual activities occurring in a person's life. A student recorded hers and found that her physical cycle was 23 days long, e.g. every twenty-three days her strength, energy, endurance and resistance to disease varied from neutral to a positive peak, down to neutral then followed by a negative peak, before rising back to neutral. Her emotional and intellectual cycles followed the same pattern, but their peaks happened five days and ten days later, respectively.

1 For the three cycles, write $P(t), E(t)\text{and } I(t)$ with t the time in days as:

a sinusoidal functions **b** $\mathrm{Im}\left(r\ \mathrm{e}^{i(c\theta+d}\right)$

2 Graph your result

3 Explain why the sum of $P(t), E(t)\text{and } I(t)$ can help determine good and bad days.

4 **a** Show that the overall physical and emotional stamina can be modelled by:

$$P(t), +E(t) = \mathrm{Im}\left(\left(\mathrm{e}^{i\left(\frac{2\pi t}{23}\right)}\right)\left(1-\mathrm{e}^{-i\pi\left(\frac{10}{23}\right)}\right)\right)$$

b Estimate it as a sine function

c Graph your results

d Find the first two best day(s)

5 Similarly, write the following as

a $\mathrm{Im}\left(r\ \mathrm{e}^{ic\theta}\right)$ **b** sinusoidal functions:

i overall physical and intellectual stamina

ii overall emotional and intellectual stamina

iii overall physical, emotional and intellectual stamina

6 Graph your results

7 For each of the above find**:**

a the best days **b** the worst days

8 Biorhythms are usually described following cycle lengths of 23, 28 and 33 for $P(t), E(t)\text{and } I(t)$ respectively. Show that $P(t) + E(t) + I(t)$ cannot be modelled by $Im(r\ cis\ \theta)$

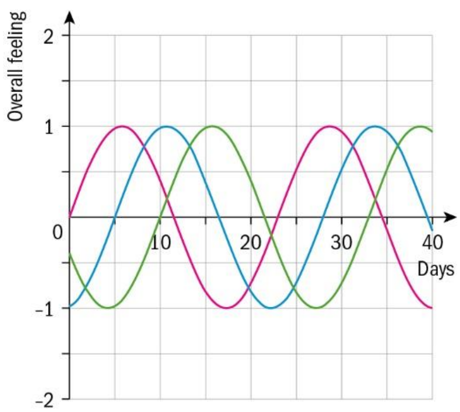
9 Leap years in happen in 2000, 2004, 2008, 2012, 2016 and 2020; hence, plot your biorhythm where t = 0 is your birth date, and find when your best days should be.

Answers

1 **a** $P(t) = \sin\left(\frac{2\pi t}{23}\right), E(x) = 2\sin\left(\frac{2\pi(t-5)}{23}\right)$ and $I(x) = 3\sin\left(\frac{2\pi(t-10)}{23}\right)$

b $P(t) = \operatorname{Im}\left(e^{i\left(\frac{2\pi t}{23}\right)}\right), E(x) = \operatorname{Im}\left(2e^{i\left(\frac{2\pi t}{23}-\frac{10\pi}{23}\right)}\right)$ and $I(x) = \operatorname{Im}\left(3e^{i\left(\frac{2\pi t}{23}-\frac{20\pi}{23}\right)}\right)$

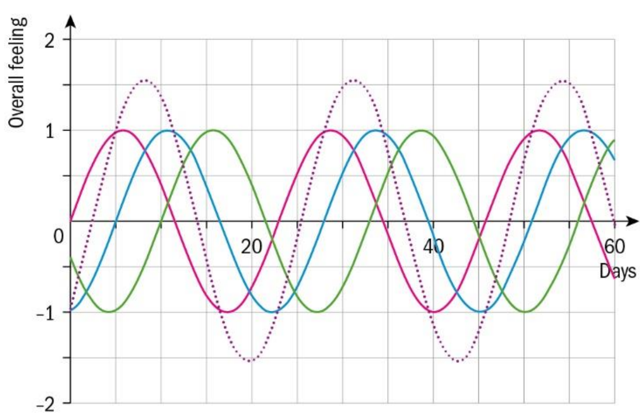
2 https://www.desmos.com/calculator/dgtwsmmhsq



3 The maximum sum of positive feelings will indicate the best day and inversely the minimum will indicate the worst.

4 **a** $P(t) + E(t) = \operatorname{Im}\left(\left(e^{i\left(\frac{2\pi t}{23}\right)}\right) + \left(\frac{e^{i\left(\frac{2\pi t}{23}\right)}}{e^{i\left(\frac{10\pi}{23}\right)}}\right)\right) = \operatorname{Im}\left(\left(e^{i\left(\frac{2\pi t}{23}\right)}\right)\left(1 + e^{-\left(\frac{10\pi}{23}\right)i}\right)\right)$

b $P(t) + E(t) = 1.55\sin\left(\frac{2\pi t}{23}t - 0.68\right) = 2\cos\left(-\frac{5\pi}{23}\right)\sin\left(\frac{2\pi}{23}t - \frac{5\pi}{23}\right)$



c

https://www.desmos.com/calculator/etqrzfl34d

d 8th and 31st

5 **i** $P(t) + I(t) = \operatorname{Im}\left(\left(e^{i\left(\frac{2\pi t}{23}\right)}\right)\left(1 + e^{-\left(\frac{20\pi}{23}\right)i}\right)\right) = 0.41;\ \sin\left(\frac{2\pi t}{23} - 1.37\right) = 2\cos\left(\frac{10\pi}{23}\right)\sin\left(\frac{2\pi}{23}t - \frac{10\pi}{23}\right)$

ii $E(t) + I(t) = \operatorname{Im}\left(\left(e^{i\left(\frac{2\pi t}{23}\right)}\right)\left(e^{-\left(\frac{10\pi}{23}\right)i} + e^{-\left(\frac{20\pi}{23}\right)i}\right)\right) = 1.551;\ \sin\left(\frac{2\pi t}{23} - 2.05\right) = 2\cos\left(\frac{5\pi}{23}\right)\sin\left(\frac{2\pi t}{23} - \frac{15\pi}{23}\right)$

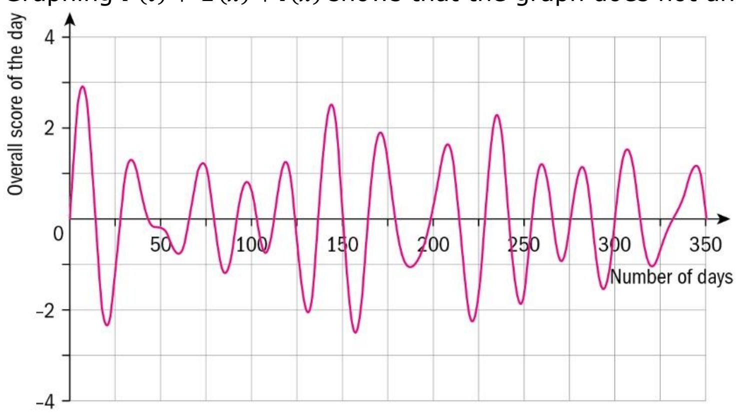
iii $P(t)+E(t)+\ I(t)=\ \mathrm{Im}\left(\left(e^{i\left(\frac{2\pi t}{23}\right)}\right)\left(1+e^{-\left(\frac{10\pi}{23}\right)i}+e^{-\left(\frac{20\pi}{23}\right)i}\right)\right)=1.407;\sin\left(\frac{2\pi t}{23}-1.37\right)=$ $1.407\sin\left(\frac{2\pi t}{23}-\frac{10\pi}{23}\right)$

6 https://www.desmos.com/calculator/ss7rgoql0a

7

	a best	**b** worse
$P(t)+\ I(x)$:	10th day and 33rd day	22th day and 45th day
$E(t)+\ I(t)$	13th day and 36th day	11th day and 24th day
$P(t)+E(t)+\ I(t)$	10th day and 33rd day	22nd and 45th day

8 Graphing $P(t)+\ E(x)+I(x)$ shows that the graph does not always have the same amplitude



https://www.desmos.com/calculator/gdr0u0u8g5

or $P(t)+\ E(x)+I(x)=\mathrm{Im}\left(e^{i\left(\frac{2\pi t}{23}\right)}\right)+\mathrm{Im}\left(e^{i\left(\frac{2\pi t}{28}\right)}\right)+\mathrm{Im}\left(e^{i\left(\frac{2\pi t}{33}\right)}\right)$ as $e^{i\left(\frac{2\pi t}{28}\right)}=e^{i\left(\frac{2\pi t}{23}\right)-i\left(\frac{5\pi t}{644}\right)}$ it is impossible to factorise in a form $\mathrm{Im}\left((1+e^{a}+e^{b})e^{i\left(\frac{2\pi t}{23}\right)}\right)$ where a and b are constants.

9 Students' own answers.

9.1 Introduction to matrices and matrix operations

1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}$.

Evaluate the following. If they do not exist then say so.
First do the calculations by hand and then check your answers using the calculator.

a $\mathbf{A} + \mathbf{B}$
b $\mathbf{B} + \mathbf{A}$
c $\mathbf{A} + \mathbf{C}$
d $4\mathbf{C}$
e $3\mathbf{A} + 2\mathbf{B}$

2 Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

a Find $\mathbf{A} + \mathbf{B}$
b Find $\mathbf{B} + \mathbf{A}$
c Comment on the validity of $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for 2×2 matrices.

Let λ be a real number.

d Find $\lambda(\mathbf{A} + \mathbf{B})$

e Find $\lambda\mathbf{A} + \lambda\mathbf{B}$

f Comment on the validity of $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$ for 2×2 matrices.

Let μ be another real number.

g Find $\lambda\mathbf{A} + \mu\mathbf{A}$

h Find $(\lambda + \mu)\mathbf{A}$

i Comment on the validity of $\lambda\mathbf{A} + \mu\mathbf{A} = (\lambda + \mu)\mathbf{A}$ for 2×2 matrices.

Answers

1	**a**	$\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$	**b**	$\begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix}$	**c**	not possible
	d	$\begin{pmatrix} 4 & 0 & 12 \\ 8 & 4 & -4 \end{pmatrix}$	**e**	$\begin{pmatrix} 5 & 4 \\ 13 & 8 \end{pmatrix}$		
2	**a**	$\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$	**b**	$\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$	**c**	always true
	d	$\begin{pmatrix} \lambda a+\lambda e & \lambda b+\lambda f \\ \lambda c+\lambda g & \lambda d+\lambda h \end{pmatrix}$	**e**	$\begin{pmatrix} \lambda a+\lambda e & \lambda b+\lambda f \\ \lambda c+\lambda g & \lambda d+\lambda h \end{pmatrix}$	**f**	always true
	g	$\begin{pmatrix} \lambda a+\mu a & \lambda b+\mu b \\ \lambda c+\mu c & \lambda d+\mu d \end{pmatrix}$	**h**	$\begin{pmatrix} \lambda a+\mu a & \lambda b+\mu b \\ \lambda c+\mu c & \lambda d+\mu d \end{pmatrix}$	**i**	always true

9.2 Matrix multiplication and properties

1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}$.

Evaluate the following. If they do not exist then say so.
First do the calculations by hand and then check your answers using the calculator.

a **AB**
b **BA**
c **AC**
d **CA**

2 A car firm has 4 factories A, B, C and D and makes cars of types 1, 2 and 3. The number of cars made by each factory every day is given in the table below;

Factory\Type	1	2	3
A	10	20	25
B	5	10	10
C	15	0	10
D	0	0	40

The number of man-hours (in hundreds) and the cost of materials (in thousands of pounds) needed to produce each type of car are given in the table below:

Type\Needs	hours	£
1	2	5
2	3	6
3	4	10

Construct a similar table that will inform the manager of each factory about the number of man-hours and material costs that will be required each day.

.

3 Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$ and let $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ be the zero matrix.

Evaluate:

a $\mathbf{AB}$
b $\mathbf{BA}$
c $\mathbf{A} \times \mathbf{O}$
d $\mathbf{O} \times \mathbf{B}$
e $\mathbf{O} \times \mathbf{O}$

Comment on the validity in general for square matrices of

f $\mathbf{C} = \mathbf{O}$ and $\mathbf{D} = \mathbf{O}$ implies $\mathbf{CD} = \mathbf{O}$

g $\mathbf{C} = \mathbf{O}$ or $\mathbf{D} = \mathbf{O}$ implies $\mathbf{CD} = \mathbf{O}$

h $\mathbf{CD} = \mathbf{O}$ implies $\mathbf{C} = \mathbf{O}$ and $\mathbf{D} = \mathbf{O}$

i $\mathbf{CD} = \mathbf{O}$ implies $\mathbf{C} = \mathbf{O}$ or $\mathbf{D} = \mathbf{O}$

Answers

1 **a** $\begin{pmatrix} 5 & -5 \\ 11 & -11 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}$ **c** $\begin{pmatrix} 5 & 2 & 1 \\ 11 & 4 & 5 \end{pmatrix}$

d not possible

2 Multiplying matrices together:

Factory\Needs	hours	£
A	180	420
B	80	185
C	70	175
D	160	400

3 **a** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

e $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **f** true **g** true **h** not true

i not true

9.3 Solving systems of equations using matrices

1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}$.

Evaluate the following. If they do not exist then say so.
First do the calculations by hand and then check your answers using the calculator.

a $\det \mathbf{A}$
b $\det \mathbf{B}$
c $\det \mathbf{C}$
d $\mathbf{A}^{-1}$
e $\mathbf{B}^{-1}$

2 **a** Write down the inverse of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 1 & 5 \end{pmatrix}$ giving the values of the entries as fractions.

b Hence solve the simultaneous equations

$$\begin{aligned} x + 2y + z &= -2 \\ 3x + y + 4z &= 6 \\ x + y + 5z &= 1 \end{aligned}$$

giving the answers as fractions.

3 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ is going to be used to code a message. The 26 letters in the English language are each converted into a 2×1 matrix as follows:

$$a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \ldots, i = \begin{pmatrix} 0 \\ 9 \end{pmatrix}, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ldots, t = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ldots, z = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

A n-lettered word is thus expressed as a $2 \times n$ matrix which is then pre-multiplied by $\mathbf{A}$ to give the encoded version as a $2 \times n$ matrix.

For example $\quad dog \to \begin{pmatrix} 0 & 1 & 0 \\ 4 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} 4 & 7 & 7 \\ -4 & -4 & -7 \end{pmatrix}$

a Encode the word *maths*.

b Decode the matrix $\begin{pmatrix} 3 & 7 & 10 & 10 & 5 & 3 & 4 \\ -3 & -4 & -7 & -7 & -5 & -3 & 2 \end{pmatrix}$

c Write down the matrix that is used in the decoding process giving the answers in fractions.

d In general state what property the coding matrix $\mathbf{A}$ would have to have for such a method to work.

Answers

1 **j** -2 **k** 0 **l** not possible

m $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$ **n** not possible

2 **a** $\begin{pmatrix} \frac{-1}{19} & \frac{9}{19} & \frac{-7}{19} \\ \frac{11}{19} & \frac{-4}{19} & \frac{1}{19} \\ \frac{-2}{19} & \frac{-1}{19} & \frac{5}{19} \end{pmatrix}$

b $\begin{pmatrix} \frac{-1}{19} & \frac{9}{19} & \frac{-7}{19} \\ \frac{11}{19} & \frac{-4}{19} & \frac{1}{19} \\ \frac{-2}{19} & \frac{-1}{19} & \frac{5}{19} \end{pmatrix}\begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$ gives $x = \frac{49}{19}, y = \frac{-45}{19}, z = \frac{3}{19}$

3 **a** $\mapsto \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 3 & 1 & 0 & 8 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 5 & 1 & 4 & 8 & 11 \\ -2 & -1 & 2 & -8 & -8 \end{pmatrix}$

b $\mapsto \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 3 & 5 & 8 & 8 & 5 & 3 & 0 \end{pmatrix} \mapsto$ *correct*

c $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{pmatrix}$

d det $\mathbf{A} \neq 0$ as $\mathbf{A}^{-1}$ must exist.

9.4 Transformations of the plane

1 Points in the plane are rotated anti-clockwise about the origin through 60° and then translated by +2 in the *x*-direction and +3 in the *y*-direction. This transformation is given by the equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.

a Find the exact values of the constants a, b, c, d, e, f.

b Find the exact image of the point $(2, 4)$.

c Find the exact point that has an image of $\left(4 + 3\sqrt{3}, 2\sqrt{3}\right)$

Answers

1 **a** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-2\sqrt{3} \\ 5+\sqrt{3} \end{pmatrix}$

Image is $\left(3-2\sqrt{3}, 5+\sqrt{3}\right)$

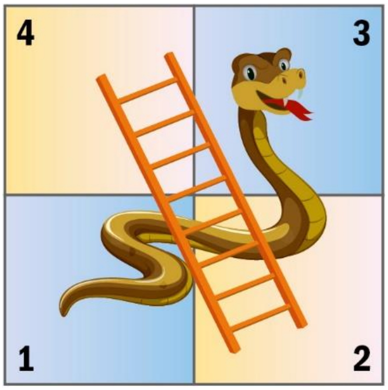
c $\begin{pmatrix} 4+3\sqrt{3} \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+3\sqrt{3} \\ -3+2\sqrt{3} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2+3\sqrt{3} \\ -3+2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

So original point was $(4, -6)$

9.5 Representing systems

10 A very elementary snakes and ladders game has a board as shown in the diagram below.



Jack, a three-year-old boy, plays this game, starting on square 1. He throws a fair coin with 1 on one side and 2 on the other side. He then moves ahead that many squares. If he lands on square 2 he must go up the ladder to the finish at square 4. If he lands on square 3 he must slide down the snake to square 1. Once arriving at square 4 he remains there.

Let the 4×1 matrix $\mathbf{S}_n = \begin{pmatrix} w_n \\ x_n \\ y_n \\ z_n \end{pmatrix}$ represent the probabilities that he is on each of the squares 1,2,3,4 respectively after he has thrown the coin n times. He starts on square 1 so $\mathbf{S}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

The transition matrix that represents this game is given by $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$,

with $\mathbf{S}_n = \mathbf{T}^n\mathbf{S}_0$

a Find:

i $\mathbf{S}_1$ **ii** $\mathbf{S}_2$ **iii** $\mathbf{S}_3$

b Conjecture a formula for $\mathbf{S}_n$.

2 A very strange country, where no-one ever dies and no-one is ever born, has three regions, N: Nuf, M: Moor and P: Plenti. Each year people move from one region to another.

The number of people (measured in thousands) in year n for each region, is given by

$\mathbf{S}_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ for the regions N, M and P respectively. Initially $\mathbf{S}_0 = \begin{pmatrix} 400 \\ 200 \\ 100 \end{pmatrix}$. The situation is

modelled by the equation $\mathbf{S}_n = \mathbf{T}^n \mathbf{S}_0$.

Each year 60% of the population of region N stay in this region and 40% move to region P.
Each year 30% of the population of region M stay in this region, 30% move to region N and 40% move to region P.
Each year 60% of the population of region P stay in this region and 40% move to region M.

a Write down the transition matrix $\mathbf{T}$.

b Find: **i** $\mathbf{S}_1$ **ii** $\mathbf{S}_2$

Answers

1 **a** **i** $\mathbf{S}_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ **ii** $\mathbf{S}_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \\ \frac{3}{4} \end{pmatrix}$ **iii** $\mathbf{S}_3 = \begin{pmatrix} \frac{1}{8} \\ 0 \\ 0 \\ \frac{7}{8} \end{pmatrix}$

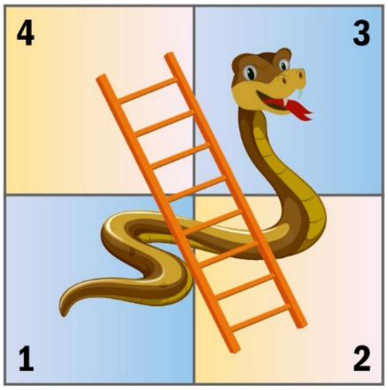
b $\mathbf{S}_n = \begin{pmatrix} \frac{1}{2^n} \\ 0 \\ 0 \\ 1-\frac{1}{2^n} \end{pmatrix}$

2 **a** $\mathbf{T} = \begin{pmatrix} 0.6 & 0.3 & 0 \\ 0 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.6 \end{pmatrix}$

b **i** $\begin{pmatrix} 300 \\ 100 \\ 300 \end{pmatrix}$ **ii** $\begin{pmatrix} 210 \\ 150 \\ 340 \end{pmatrix}$

9.6 Representing steady state systems

1 A very elementary snakes and ladders game has a board as shown in the diagram below.



Jack, a three-year-old boy, plays this game, starting on square 1. He throws a fair coin with 1 on one side and 2 on the other side. He then moves ahead that many squares. If he lands on square 2 he must go up the ladder to the finish at square 4. If he lands on square 3 he must slide down the snake to square 1. Once arriving at square 4 he remains there.

Let the 4×1 matrix $\mathbf{S}_n = \begin{pmatrix} w_n \\ x_n \\ y_n \\ z_n \end{pmatrix}$ represent the probabilities that he is on each of the squares 1,2,3,4 respectively after he has thrown the coin n times. He starts on square 1 so

$$\mathbf{S}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The transition matrix that represents this game is given by $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$,

with $\mathbf{S}_n = \mathbf{T}^n\mathbf{S}_0$

a Find:

i $\mathbf{S}_1$ **ii** $\mathbf{S}_2$ **iii** $\mathbf{S}_3$

b Conjecture a formula for $\mathbf{S}_n$.

c Let $\lim_{n\to\infty} \mathbf{S}_n = \mathbf{S}$. Find $\mathbf{S}$.

d Show that $\mathbf{S}$ is a steady state solution to the matrix recurrence relation.

e Explain why having a steady state solution must imply that matrix $\mathbf{T}$ has an eigenvalue equal to 1.

f Check this using the calculator to find $|\mathbf{T}-\mathbf{I}|$.

Let U be the discrete random variable representing the number of times that Jack has to throw the coin until he lands on square 4.

g Find

i $P(U=1)$ **ii** $P(U=2)$ **iii** $P(U=3)$

h State the type of progression that these probabilities follow.

i Use part (h) to show that the sum of this infinite number of probabilities equals 1.

Let $E(U)$ be the expected number of times that Jack will have to throw the coin in order to reach square 4.

j Write down the start of an infinite series for $E(U)$.

k Multiply the expression found in part (j) by $\frac{1}{2}$ and subtract from the original expression to show that $E(U)=2$

2 A very strange country, where no-one ever dies and no-one is ever born, has three regions, N: Nuf, M: Moor and P: Plenti. Each year people move from one region to another.

The number of people (measured in thousands) in year n for each region, is given by $\mathbf{S}_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ for the regions N, M and P respectively. Initially $\mathbf{S}_0 = \begin{pmatrix} 400 \\ 200 \\ 100 \end{pmatrix}$. The situation is modelled by the equation $\mathbf{S}_n = \mathbf{T}^n\mathbf{S}_0$.

Each year 60% of the population of region N stay in this region and 40% move to region P.
Each year 30% of the population of region M stay in this region, 30% move to region N and 40% move to region P.
Each year 60% of the population of region P stay in this region and 40% move to region M.

a Write down the transition matrix $\mathbf{T}$.

b Find: **i** $\mathbf{S}_1$ **ii** $\mathbf{S}_2$

c Find the steady state solution by solving three simultaneous equations.

d Check the plausibility of your answer in part (c) by finding $\mathbf{S}_{50}$ using a calculator.

Answers

1 **a** **i** $\mathbf{S}_1=\begin{pmatrix}\frac{1}{2}\\0\\0\\\frac{1}{2}\end{pmatrix}$ **ii** $\mathbf{S}_2=\begin{pmatrix}\frac{1}{4}\\0\\0\\\frac{3}{4}\end{pmatrix}$ **iii** $\mathbf{S}_3=\begin{pmatrix}\frac{1}{8}\\0\\0\\\frac{7}{8}\end{pmatrix}$

b $\mathbf{S}_n=\begin{pmatrix}\frac{1}{2^n}\\0\\0\\1-\frac{1}{2^n}\end{pmatrix}$

c $\mathbf{S}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

d $\mathbf{TS}=\mathbf{S}$ is true

e $\mathbf{TS}=1\times\mathbf{S}$ shows that 1 is an eigenvalue with eigenvector of $\mathbf{S}$.

f calculator gives value of $|\mathbf{T}-\mathbf{I}|$ as 0.

g **i** $\frac{1}{2}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{8}$

h Geometric progression with $a=\frac{1}{2}$ and $r=\frac{1}{2}$

i $S_\infty=\frac{a}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$

j $\mathrm{E}(U)=1\times\frac{1}{2}+2\times\frac{1}{4}+3\times\frac{1}{8}+4\times\frac{1}{16}+\ldots$

k $\frac{\mathrm{E}(U)}{2}=1\times\frac{1}{4}+2\times\frac{1}{8}+3\times\frac{1}{16}+\ldots$

Subtracting $\frac{\mathrm{E}(U)}{2}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1\Rightarrow\mathrm{E}(U)=2$

2 **a** $\mathbf{T}=\begin{pmatrix}0.6 & 0.3 & 0\\ 0 & 0.3 & 0.4\\ 0.4 & 0.4 & 0.6\end{pmatrix}$

b **i** $\begin{pmatrix}300\\100\\300\end{pmatrix}$ **ii** $\begin{pmatrix}210\\150\\340\end{pmatrix}$

c Solving $\begin{aligned}0.6x + 0.3y \quad &= x\\ 0.3y + 0.4z &= y\\ x + y + z &= 700\end{aligned}$ gives steady state of $\begin{pmatrix}150\\200\\350\end{pmatrix}$

d calculator gives $\mathbf{S}_{50}$ as $\begin{pmatrix}150\\200\\350\end{pmatrix}$

9.7 Eigenvalues and eigenvectors

1 Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

a Show that the eigenvectors of $\mathbf{A}$ satisfy the equation

$$\lambda^2 + p\lambda + q = 0$$

where p and q are functions of a, b, c, d that are to be determined.

b Verify that
$\mathbf{A}^2 + p\mathbf{A} + q\mathbf{I} = \mathbf{O}$.

c Hence given that $\mathbf{A}$ is non-singular show that
$\mathbf{A}^{-1} = \frac{-1}{q}(\mathbf{A} + p\mathbf{I})$.

2 Let $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$.

a Find the eigenvalues for matrix $\mathbf{A}$.
b For each eigenvalue find the corresponding eigenvectors.
c Hence express $\mathbf{A}$ as $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix.
d Use part (c) to evaluate $\mathbf{A}^{10}$.
e Confirm your answer by finding $\mathbf{A}^{10}$ directly using the calculator.
f Use part (d) to evaluate $\mathbf{A}^{11}$.
g Confirm your answer by finding $\mathbf{A}^{11}$ directly using the calculator.

h Use part (c) to evaluate $\mathbf{A}^{2n}$ where $n \in \mathbb{Z}^+$.

i Use part (h) to evaluate $\mathbf{A}^{2n+1}$ where $n \in \mathbb{Z}^+$.

j Using part (h), state what the case with $n = 1$ tells you about $\mathbf{A}^{-1}$.
Confirm this:

i using the formula for $\mathbf{A}^{-1}$
ii using the calculator.

3 Let $\mathbf{A} = \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$.
b For each eigenvalue find the corresponding eigenvectors.
c Show that eigenvectors corresponding to different eigenvalues are perpendicular.

d Hence express $\mathbf{A}$ as $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ represents a rotational matrix through an angle of θ degrees anticlockwise about the origin, $0^\circ < \theta < 180^\circ$.

e Find the value of θ.

Answers

1 **a** $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0 \Rightarrow \lambda^2 + (-a-d)\lambda + (ad-bc) = 0$

b $\mathbf{A}^2 + p\mathbf{A} + q\mathbf{I} =$

$$\begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

c $\mathbf{A}(\mathbf{A}+p\mathbf{I}) = -q\mathbf{I} \Rightarrow \mathbf{A} \times \dfrac{-1}{q}(\mathbf{A}+p\mathbf{I}) = \mathbf{I} \ \Rightarrow \mathbf{A}^{-1} = \dfrac{-1}{q}(\mathbf{A}+p\mathbf{I})$

2 **a** $\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow (\lambda-1)(\lambda+1) = 0$

$\lambda = 1$ or -1

b $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = 3y$

For $\lambda = 1$ eigenvectors have the form $\begin{pmatrix} 3k \\ k \end{pmatrix}$

$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = y$

For $\lambda = -1$ eigenvectors have the form $\begin{pmatrix} k \\ k \end{pmatrix}$

c $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

d $\mathbf{A}^{10} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{10}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e calculator confirms $\mathbf{A}^{10} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

f $\mathbf{A}^{11} = \mathbf{A}^{10} \times \mathbf{A} = \mathbf{A}$

g calculator confirms $\mathbf{A}^{11} = \mathbf{A}$

h $\mathbf{A}^{2n} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{2n}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{n}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i $\mathbf{A}^{2n+1} = \mathbf{A}^{2n} \times \mathbf{A} = \mathbf{A}$

j $\mathbf{A}^2 = \mathbf{I}$ hence $\mathbf{A}^{-1} = \mathbf{A}$

i det $\mathbf{A} = -4 + 3 = -1\theta$, applying formula $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

ii calculator confirms $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

3 **a** $\begin{vmatrix} 1-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4 \text{ or } -1$

b $\lambda = 4,\ \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \sqrt{6}y = 3x \Rightarrow y = \frac{\sqrt{3}}{\sqrt{2}}x$

Eigenvectors are of the form $\begin{pmatrix} \sqrt{2}t \\ \sqrt{3}t \end{pmatrix}$

$\lambda = -1,\ \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow -\sqrt{6}y = 2x \Rightarrow y = -\frac{\sqrt{2}}{\sqrt{3}}x$

Eigenvectors are of the form $\begin{pmatrix} -\sqrt{3}s \\ \sqrt{2}s \end{pmatrix}$

c $\begin{pmatrix} \sqrt{2}t \\ \sqrt{3}t \end{pmatrix} \bullet \begin{pmatrix} -\sqrt{3}s \\ \sqrt{2}s \end{pmatrix} = 0$ so the vectors are perpendicular

d Choosing eigenvectors of length 1

$$\begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{-\sqrt{3}}{\sqrt{5}} \\ \frac{\sqrt{3}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{-\sqrt{3}}{\sqrt{5}} \\ \frac{\sqrt{3}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{-\sqrt{3}}{\sqrt{5}} \\ \frac{\sqrt{3}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{-\sqrt{3}}{\sqrt{5}} \\ \frac{\sqrt{3}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}^{-1}$$

e $\cos\theta = \frac{\sqrt{2}}{\sqrt{5}} \Rightarrow \theta = 50.8^\circ\ (3\ sf)$

10.1 Limits and derivatives

1 Find the derivatives of the following functions:

a $f\left(x\right)=6x^4-3x^2+2$

b $f\left(x\right)=2x^2-\frac{7}{\sqrt{x}}$

c $f\left(x\right)=\frac{3}{x}+6$

d $f\left(x\right)=6x^3(2x-\frac{1}{x^5})$

2 **a** Find the derivative of $y=3x+\frac{12}{x}$.

b Determine if the function is increasing or decreasing when $x=4$.

c Find the range of values for which the function is increasing.

3 The volume of a sphere is given by the formula $V=\frac{4}{3}\pi r^3$.

a Find $\frac{\mathrm{d}V}{\mathrm{d}r}$ and explain what it represents.

b Find $\frac{\mathrm{d}V}{\mathrm{d}r}$ when $r = 2$.

4 The profit, in euros, from manufacturing roller skates can be modelled by the function $P(x)=-0.00009375x^3+0.06x^2-6.5625x+600$ where x is the number of roller skates produced.

a Find the cost when no roller skates are produced.

b Find $\frac{\mathrm{d}P}{\mathrm{d}x}$ and explain what this represents.

c Find $\frac{\mathrm{d}P}{\mathrm{d}x}$ when $x = 50$, $x = 200$ and $x = 400$ and comment on these results.

5 The gradient of the curve $y=2x^2-\frac{16}{\sqrt{x}}$ at point A is 17.

Find the coordinates of point A.

6 Find the equation of the tangent and the equation of the normal to the curve $y=x^2+\frac{4}{x}+3$ at the point where $x = 1$.

7 The gradient of the tangent to the curve $y = x^3 + 2ax + 3$ at the point (1, y) is 7.

Find the values of a and y.

8 Using your GDC, find the gradient of the following curves at the point where $x = 2$.

a $f(x) = \dfrac{3x^2 + 4}{\sqrt{2x - 1}}$ **b** $f(x) = (6x + 1)^2 \times \ln(2x)$

c $f(x) = \dfrac{e^{3x} - 4}{(x + 1)^3}$

9 An open box is $4x$ cm long, $2x$ cm wide and y cm high. The volume of the box is 600 cm^3.

a Write down an expression for the surface area, S, in terms of x only.

b Find $\dfrac{dS}{dx}$.

c Hence, find the minimum surface area of the box and the value of x where this occurs.

10 The total surface area of a closed cylinder is 1000 cm^2.

Find the maximum volume and the dimensions of the cylinder when this occurs.

11 A glass is in the shape of an open cylinder. When full, the glass holds 250 cm^3.

Find the minimum possible surface area of the glass and the dimensions when this occurs.

12 The cost of publishing a poetry book is 6000 euros plus 6 euros for each book printed.

The selling price of each book is modelled by the function $s(x) = 25 - \dfrac{x}{250}$.

a Find an expression for the profit function, $P(x)$.

b Find the number of copies that should be sold in order to make the maximum profit.

Answers

1 **a** $f'(x) = 24x^3 - 6x$

b $f'(x) = 4x + \dfrac{7}{2\sqrt{x^3}}$

c $f'(x) = \dfrac{-3}{x^2}$

d $f'(x) = 48x^3 + \dfrac{12}{x^3}$

2 **a** $\dfrac{dy}{dx} = 3 - \dfrac{12}{x^2}$

b increasing

c $x > 2$ and $x < -2$

3 **a** $\dfrac{dV}{dr} = 4\pi r^2$ this is the rate of change of the volume with respect to the radius.

b 16π

4 **a** 600

b $\dfrac{dP}{dx} = -0.00028125x^2 + 0.12x - 6.5625$

c at 50 it is −1.266 and so is decreasing, at 200 it is 6.1875 and so is increasing and at 400 it is −3.5625 and so is decreasing.

5 $4x + \dfrac{16}{2\sqrt{x^3}} = 17$. Using solver, $x = 4$ and so, A is (4, 0)

6 $\dfrac{dy}{dx} = 2x - \dfrac{4}{x^2}$ when $x = 1$, $\dfrac{dy}{dx} = -2$ and $y = 8$ so, equation of tangent is

$y - 8 = -2(x - 1)$ which gives $y = -2x + 10$

The gradient of the normal is $\dfrac{1}{2}$ and so the equation of the normal is $y - 8 = \dfrac{1}{2}(x - 1)$

$y = \dfrac{1}{2}x + \dfrac{15}{2}$ or $2y = x + 15$

7 $\dfrac{dy}{dx} = 3x^2 + 2a$ when $x = 1$, $3 + 2a = 7$ giving $a = 2$ and $y = 8$

8 **a** 3.849 **b** 300.8 **c** 30.0

9 **a** $S = 8x^2 + \frac{900}{x}$

b $\frac{dS}{dt} = 16x - \frac{900}{x^2}$

c $S = 352$ when $x = 3.83$

10 V = 2430 cm^3 when $r = 7.284$ and $h = 14.566$

11 S = 174 cm^2 when $r = 4.30$ and $h = 4.30$

12 **a** $P = \left(25x - \frac{x^2}{250}\right) - (6000 + 6x)$

b $\frac{dP}{dx} = 25 - \frac{x}{125} - 6 = 0$ when $x = 2375$

10.2 Differentiation: further rules and techniques

1 Find the derivatives of the following functions:

a $y = (3x^2 - 1)^5$ **b** $y = \sin^2(3x)$ **c** $y = \ln(\frac{x^2+1}{2x-1})$

d $y = xe^{3x}$ **e** $y = e^x \tan(3x)$ **f** $y = \frac{(x^2-1)}{(2x+1)^2}$

g $y = \frac{\sin^2 x}{e^x}$ **h** $y = \frac{(\cos 2x)^2}{x}$ **i** $y = (2x-3)^4(\sin(2x))^3$

j $y = \frac{\ln(x^3)}{e^{2x}+1}$

2 Find the equation of the tangent to the curve $f(x) = 3\sin(x) + 2$ at the point where $x=0$.

3 Find the equation of the normal to the curve $f(x) = (2x^2 - 1)^3$ at the point when $x=1$.

4 The gradient of the normal to the curve $f(x) = 2e^x + x$ at the point A is $-\frac{1}{3}$.

Find the coordinates of point A.

5 The average temperature in an outhouse can be modelled by the function

$$f(t) = -6\cos\left(\frac{\pi}{15}t\right) + 14 \quad \text{for} \quad 0 \le t \le 24.$$

a Find $f'(t)$ and explain what this represents.

b Find the maximum temperature and the value of t when this occurs.

6 The path of a stone projected into the air can be modelled by the function $f(t) = e^t \sin(t)$ for $0 \le t \le 5$ where t is the time in seconds.

a Find $f'(t)$.

b Find the maximum height reached by the stone.

c Find the time when the stone hits the ground.

7 A function has equation $f(x) = \dfrac{(x+2)^2}{(x+1)}$.

The gradient of the tangent at point *A* is $\frac{3}{4}$.

Find the coordinates of point *A* given that $x > 0$.

8 An arched doorway can be modelled by the function $f\left(x\right) = 2\sin\left(x\right)$ for $0 \le x \le \pi$.

A small "peephole" is placed at the point where the normals to the curve at the points

$x = \dfrac{\pi}{3}$ and $x = \dfrac{2\pi}{3}$ meet.

Find the coordinates of the "peephole".

9 A right circular cone has radius, *r*, and height, *h*.

a Find an expression for the slant height, *s*.

b Find the maximum possible volume of a cone with a slant height of 12.

Answers

1 **a** $30x\left(3x^2-1\right)^4$ **b** $6\sin(3x)\cos(3x)$ **c** $\dfrac{2x}{x^2+1}-\dfrac{2}{2x-1}$

d $e^{3x}(3x+1)$ **e** $3e^x\sec^2 3x+e^x\tan 3x$ **f** $\dfrac{2(x+2)}{(2x+1)^3}$

g $\dfrac{\sin x(2\cos x-\sin x)}{e^x}$ **h** $\dfrac{-\cos 2x(4x\sin 2x+\cos 2x)}{x^2}$

i $2(2x-3)^3(\sin 2x)^2[3(2x-3)\cos 2x+4\sin 2x]$ **j** $\dfrac{3\left(e^{2x}+1\right)-6xe^{2x}\ln x}{x\left(e^{2x}+1\right)^2}$

3 $12y+x=13$

4 gradient of tangent = 3 so, $f(x)=2e^x+1=3$ which gives x = 0 and y = 2.

5 **a** $f(t)=6\dfrac{\pi}{15}\sin\left(\dfrac{\pi}{15}t\right)$ the rate of change of the temperature with respect to time.

b 20 when t = 15

6 **a** $f(t)=e^t\cos t+e^t\sin t$

b 7.46 m when t = 2.36 s

c t = 3.14 s

7 $f(x)=\dfrac{x(x+2)}{(x+1)^2}=\dfrac{3}{4}$ gives x = 1 and y = 4.5

8 $f(x)=2\cos x$

When $x=\dfrac{\pi}{3}$, y = 1.732 and $f(x)=1$ so the gradient of the normal is −1 and its equation is

$$y-1.732=-1\left(x-\frac{\pi}{3}\right)$$

When $x=\dfrac{2\pi}{3}$, y = 1.732 and $f(x)=-1$ so the gradient of the normal is 1 and its equation is

$$y-1.732=1\left(x-\frac{2\pi}{3}\right)$$

Solving simultaneously gives $x = 1.57$ and $y = 1.21$

9 **a** $s = \sqrt{r^2 + h^2}$

b $h = \sqrt{\left(144 - r^2\right)}$ so $V = \frac{1}{3}\pi r^2(\sqrt{\left(144 - r^2\right)}\ )$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r\left(288 - 3r^2\right)}{3\sqrt{144 - r^2}} = 0 \text{ at maximum or minimum}$$

So, $V = 696$ when $r = 9.80$

10.3 Applications and higher derivatives

1 $y=-x^3-4x^2+16x+100$.

Find

a $\frac{dy}{dx}$

b the coordinates of the points on the curve where $\frac{dy}{dx}=0$.

c $\frac{d^2y}{dx^2}$

d Hence, determine whether the points found in part b are local maxima or local minima.

e Find the coordinates of the point of inflection.

2 $y=4x^5+2.5x^4-10x^3$.

Find

a $\frac{dy}{dx}$

b the coordinates of the points on the curve where $\frac{dy}{dx}=0$.

c $\frac{d^2y}{dx^2}$

d Hence, determine whether the points found in part b are local maxima or local minima.

e Find the coordinates of the point of inflection.

3 The growth of a sunflower can be modelled by the equation $f(x)=\frac{2.06}{1+20.7e^{-0.555x}}$

Find $f'(x)$ and explain what this represents.

4 A hot air balloon is rising vertically. Kristiaan is standing 10 metres horizontally from the point on the ground where the balloon took off.

Find the rate at which the balloon is rising when the angle between Kristiaan and the balloon is 30° and is increasing at a rate of 2° per second.

5 When the radius of a circle is 5 cm, its area is increasing at a rate of 10π cm^2/s.

Find the rate at which the radius is increasing.

6 The distance that a car has travelled after t seconds can be modelled by the function

$f(t) = 6t^2 + 80$ where t is the time in seconds.

a Find an expression for the velocity of the car at time t.

b Find an expression for the acceleration of the car at time t.

7 Harry is standing at the top of a building that is 100 metres high. He throws a ball into the air. The path of the ball can be modelled by the equation $f(t) = -16t^2 + 50t + 102$ where t is the time in seconds.

a Find an expression for the velocity of the ball.

b Find the value of t and $f(t)$ when the velocity is equal to zero and explain what this point represents.

c Find the time when the ball first hits the ground.

Answers

1 **a** $\frac{dy}{dx} = -3x^2 - 8x + 16$

b $x = -4 \text{ and } \frac{4}{3}$

c $\frac{d^2y}{dx^2} = -6x - 8$

d When $x = -4$, $\frac{d^2y}{dx^2} = 16$ and is a minimum

When $x = \frac{4}{3}$, $\frac{d^2y}{dx^2} = -16$ and is a maximum

e $-6x - 8 = 0$ gives $x = \frac{-4}{3}$ and y = 73.9

2 **a** $\frac{dy}{dx} = 20x^4 + 10x^3 - 30x^2$

b x = -1.5, 0 and 1; y = 16, 0 and -3.5

c $\frac{d^2y}{dx^2} = 80x^3 + 30x^2 - 60x$

d when $x = -1.5$, $\frac{d^2y}{dx^2} = -112.5$ and is a maximum

when $x = 1$, $\frac{d^2y}{dx^2} = 50$ and is a minimum

when $x = 0$, $\frac{d^2y}{dx^2} = 0$ and it is undecided

e $\frac{d^2y}{dx^2} = 0$ gives $x = -1.07, 0.699$ and 0

So, points of inflection are (0, 0), (-1.07, 9.92) and (0.699, -2.15)

3 $f(x) = \frac{2.06\left(11.4885e^{-0.555x}\right)}{(1 + 20.7e^{-0.555x})^2}$ this represents the rate of change of the growth with respect to time.

4 $\frac{d\theta}{dt} = 2$ Let the height of the balloon be h and the angle between Kristiaan and the balloon be θ

$h = 10\tan\theta$ so, $\dfrac{dh}{d\theta} = 10\sec^2\theta$

$$\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt} = 20\sec^2\theta$$

When $\theta = 30°$, $\dfrac{dh}{dt} = 20\sec^2 30 = \dfrac{80}{3}$

5 $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = 10\pi$$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times 10\pi = \frac{5}{r}$$

When $r = 5$, $\dfrac{dr}{dt} = 1$

6 **a** velocity = $12t$ **b** acceleration = 12

7 **a** $v = -32t + 50$

b $t = \dfrac{50}{32} = 1.5625$ and $f(t) = 141$ – this is the maximum height of the ball and the time it occurred.

c 4.53 s

11.1 Finding approximate areas for irregular regions

1 Evaluate the following definite integrals.

First do this manually and then check your answers with the calculator.

a $\int_0^1 4x^3 \, dx$

b $\int_1^2 x^2 - 3 \, dx$

c $\int_0^4 \sqrt{x} \, dx$

d $\int_1^2 x + \frac{1}{x^2} \, dx$

e $\int_0^{16} \sqrt[4]{x} \, dx$

2 **a** Find an approximation to the definite integral $\int_0^{\pi} \sin x \, dx$ using the Trapezium rule with a step width of $\frac{\pi}{6}$. Give your answer to three decimal places.

b Find the exact value of $\int_0^{\pi} \sin x \, dx$. Do this manually and then check the answer with the calculator.

c Find the absolute percentage error made when using the Trapezium rule giving the answer to 1 decimal place.

d Explain why the Trapezium rule underestimated the area.

3 An even function is one for which $f(-x) = f(x), \forall x \in \mathbb{R}$

An odd function is one for which $f(-x) = -f(x), \forall x \in \mathbb{R}$

Let $g(x)$ be an even function, $h(x)$ be an odd function and k be a positive constant.

a Find, with an explanation, the value of $\int_{-k}^{k} h(x) \, dx$.

b If $\int_{-k}^{k} g(x) \, dx = 20$, find, with an explanation, the value of $\int_0^{k} g(x) \, dx$.

c Find, with an explanation, the value of $\int_{-k}^{k} h(x) g(x) \, dx$.

d Find, with an explanation, the value of $h(0)$.

Answers

1 **a** $\left[x^4\right]_0^1 = 1$ **b** $\left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = \frac{-2}{3}$

c $\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^4 = \frac{16}{3}$ **d** $\left[\frac{x^2}{2} - x^{-1}\right]_1^2 = \left(2 - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right) = 2$

e $\left[\frac{4}{5}x^{\frac{5}{4}}\right]_0^{16} = \frac{128}{5}$

2 **a** $\frac{\pi}{12}\left(0 + 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) + 0\right) = 1.954\,(3\text{ dp})$

b $\left[-\cos x\right]_0^{\pi} = 2$

c $\frac{|2 - 1.954|}{2} \times 100\% = 2.3\%\,(1\text{ dp})$

d underestimates as the curve is concave down.

3 **a** 0 since $\int_{-k}^{0} h(x)\,\mathrm{d}x = -\int_{0}^{k} h(x)\,\mathrm{d}x$

b 10 since $\int_{-k}^{0} g(x)\,\mathrm{d}x = \int_{0}^{k} g(x)\,\mathrm{d}x$

c 0 since $h(x)g(x)$ is an odd function

d $h(0) = h(-0) = -h(0) \Rightarrow 2h(0) = 0 \Rightarrow h(0) = 0$

11.2 Indefinite integrals and techniques of integration

1 Find the following indefinite integrals.

a $\int x^2 + x \, dx$

b $\int x^3 + x + 1 \, dx$

c $\int 3x^2 + 4x + 5 \, dx$

d $\int ax^2 + bx + c \, dx$

e $\int x^{99} - x^{49} \, dx$

f $\int x^{-5} + x^{-3} \, dx$

g $\int \frac{1}{x^2} + 7 \, dx$

h $\int \sqrt{x} + \sqrt[3]{x} \, dx$

i $\int \frac{1}{\sqrt{x}} + \frac{1}{x^5} \, dx$

4 a Find the following indefinite integrals.

i $\int \cos(2x) \, dx$

ii $\int \frac{1}{\cos^2 x} \, dx$

iii $\int e^{-x} dx$

iv $\int \frac{1}{x} + \frac{1}{x^2} \, dx$

b Use part (a) to manually find the exact values of the following integrals. Then check your answers with the calculator.

i $\int_0^{\frac{\pi}{4}} \cos(2x) \, dx$

ii $\int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx$

iii $\int_0^2 e^{-x} dx$

iv $\int_1^2 \frac{1}{x} + \frac{1}{x^2} \, dx$

5 Find the following indefinite integrals.

a $\int x\left(x^3 + 3\right) dx$

b $\int \frac{x^3 - 2x^2}{x}\,dx$

c $\int \sin^2 x + \cos^2 x\,dx$

d $\int \left(e^x + 1\right)^2 dx$

12 Find the following indefinite integrals. Ideally do them by inspection, but if that is not obvious employ a suitable substitution.

a $\int (3x + 4)^7\,dx$

b $\int e^{4x-1}\,dx$

c $\int \frac{1}{2x + 5}\,dx$

d $\int \frac{\cos x}{\sin x}\,dx$

e $\int \frac{x}{x^2 + 1}\,dx$

f $\int e^x \sin\left(e^x + 2\right)dx$

g $\int \sin x \cos^5 x\,dx$

h $\int \frac{\tan x}{\cos^2 x}\,dx$

i $\int \frac{1}{x(\ln x)^2}\,dx$

Answers

1 **a** $\frac{x^3}{3}+\frac{x^2}{2}+c$ **b** $\frac{x^4}{4}+\frac{x^2}{2}+x+c$ **c** x^3+2x^2+5x+c

d $\frac{a}{3}x^3+\frac{b}{2}x^2+cx+d$ **e** $\frac{x^{100}}{100}-\frac{x^{50}}{50}+c$ **f** $-\frac{x^{-4}}{4}-\frac{x^{-2}}{2}+c$

g $-x^{-1}+7x+c$ **h** $\frac{2x^{\frac{3}{2}}}{3}+\frac{3x^{\frac{4}{3}}}{4}+c$ **i** $2x^{\frac{1}{2}}-\frac{x^{-4}}{4}+c$

e $\left[\frac{4}{5}x^{\frac{5}{4}}\right]_0^{16}=\frac{128}{5}$

4 **a** **i** $\frac{1}{2}\sin(2x)+c$ **ii** $\tan x+c$

iii $-e^{-x}+c$ **iv** $\ln|x|-x^{-1}+c$

b **i** $\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}=\frac{1}{2}$ **ii** $\left[\tan x\right]_0^{\frac{\pi}{3}}=\sqrt{3}$

iii $\left[-e^{-x}\right]_0^{2}=1-e^{-2}$

iv $\left[\ln|x|-x^{-1}\right]_1^{2}=\left(\ln 2-\frac{1}{2}\right)-(-1)=\ln 2+\frac{1}{2}$

5 **a** $\int x^4+3x\,dx=\frac{x^5}{5}+\frac{3x^2}{2}+c$

b $\int x^2-2x\,dx=\frac{x^3}{3}-x^2+c$

c $\int 1\,dx=x+c$

d $\int e^{2x}+2e^x+1\,dx=\frac{e^{2x}}{2}+2e^x+x+c$

12 **a** $\frac{(3x+4)^8}{24}+c$ **b** $\frac{e^{4x-1}}{4}+c$

c $\frac{1}{2}\ln|2x+5|+c$ **d** $\ln|\sin x|+c$

e $\frac{1}{2}\ln\left(x^2+1\right)+c$ **f** $-\cos\left(e^x+2\right)+c$

g $\frac{-1}{6}\cos^6 x+c$ **h** $1/\cos^2 x + c$

i $-(\ln x)^{-1}+c$

11.3 Applications of integration

1 **a** Find the volume of the solid of revolution created when the curve $y = \sin x$ from $(0,0)$ to $\left(\frac{\pi}{2}, 1\right)$ is rotated through 2π about the x axis.

b Describe in everyday language the shape that is formed.

c Find the volume of the solid of revolution created when the same portion of curve is rotated through 2π about the y axis.

d Describe in everyday language the shape that is formed this time.

2 Find the area enclosed between the curves

$y = 2x^2 - 15x + 37$ and $y = -3x^2 + 20x - 13$.

First of all obtain the exact answer by doing the integration manually and then check your answer using the calculator.

3 A farmer owns a large field. Its boundaries are given by the equations

$x = 1, \quad x = 5, \quad y = 0, \quad y = -x^2 + bx + c$

where x and y are in units of 100 metres.

The field is symmetrical about the line $x = 3$.

a Find the value of the constant b.

The area of the field is $26\frac{2}{3} \times 10^4\ \text{m}^2$

b Find the value of the constant c.

Answers

1 **a** $\pi\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = 2.47 \, (3 \text{ sf})$ **b** a bowl

c $\pi\int_0^1 (\arcsin x)^2 \, dx = 1.47 \, (3 \text{ sf})$ **d** a funnel

2 $2x^2 - 15x + 37 = -3x^2 + 20x - 13 \Rightarrow 5x^2 - 35x + 50 = 0 \Rightarrow x^2 - 7x + 10 = 0$

$\Rightarrow (x-2)(x-5) = 0 \Rightarrow x = 2 \text{ or } 5$

$$\int_2^5 (-3x^2 + 20x - 13) - (2x^2 - 15x + 37)\, dx = \int_2^5 -5x^2 + 35x - 50 \, dx = \left[\frac{-5}{3}x^3 + \frac{35}{2}x^2 - 50x\right]_2^5$$

$$= \left(\frac{-5}{3}125 + \frac{35}{2}25 - 50 \times 5\right) - \left(\frac{-5}{3}8 + \frac{35}{2}4 - 50 \times 2\right) = 22.5$$

3 **a** axis of symmetry is $\frac{-b}{-2} = 3 \Rightarrow b = 6$

b $\int_1^5 -x^2 + 6x + c = \left[\frac{-x^3}{3} + 3x^2 + cx\right]_1^5 = \left(\frac{-125}{3} + 75 + 5c\right) - \left(\frac{-1}{3} + 3 + c\right) = 30\tfrac{2}{3} + 4c$

$= 26\tfrac{2}{3} \Rightarrow c = -1$

11.4 Finding approximate areas for irregular regions

1 A function $f(x)$ satisfies $\frac{\mathrm{d}f}{\mathrm{d}x} = x^2(x^3+1)^4$ and passes through the point $(1,2)$.

a Find the equation of $f(x)$.

b Find $f(0)$.

2 The velocity of an object moving in a straight line is given by $v = \frac{\mathrm{d}x}{\mathrm{d}t} = t^2 - 9t + 18 \ \ m/s$

where x is the displacement from the origin in metres and t is time in seconds. Initially $x = 1$.

a Find the acceleration of the object as a function of t .

b Find the times when the object is at rest.

c Find the displacement of the object as a function of t .

d Find the displacement of the object when $t = 4$

e Find the total distance travelled by the object in the first 4 seconds.

f Find the speed of the object when $t = 4$.

3 A differential equation is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = yx^2, \ y > 0 \ , \ y(0) = 3$$

a Solve this differential equation giving your answer in the form $y = y(x)$.

b Hence find $y(2)$ exactly.

c Use Euler's method to find an approximation for the value of y when $x = 2$ using a step length of 0.5 .

d Find the absolute percentage error that is made when employing Euler's method.

e State one change that could be made when using Euler's method on this differential equation to make it more accurate.

4 As time progresses a spherical stone erodes, but remains in the shape of a sphere. Let the radius of the stone be r, measured in centimetres and let time t be measured in years. The rate of change of the radius of the stone is proportional to the surface area of the stone, with constant of proportionality, k. Initially $r = 10$ but after 100 years $r = 5$.

a Write down the differential equation that describes this situation.

b Solve this differential equation to show that $r = \frac{1000}{10+t}$

c Find r after 900 years.

d Find how long it takes for r to become $\frac{1}{10}$.

e Write down the limit of r as t tends to infinity.

Answers

1 **a** $f(x) = \int x^2\left(x^3+1\right)^4 \, dx = \dfrac{\left(x^3+1\right)^5}{15} + c$

Through $(1,2) \Rightarrow c = \dfrac{-2}{15}$ $\quad f(x) = \dfrac{\left(x^3+1\right)^5}{15} - \dfrac{2}{15}$

b $f(0) = -\dfrac{1}{15}$

2 **a** $a = \dfrac{dv}{dt} = 2t - 9$

b $v = (t-6)(t-3)$ so at rest at $t = 3$ and 6

c $x = \int t^2 - 9t + 18 \; dt = \dfrac{t^3}{3} - \dfrac{9t^2}{2} + 18t + c$

$x = 1$ at $t = 0 \Rightarrow x = \dfrac{t^3}{3} - \dfrac{9t^2}{2} + 18t + 1$

d $x(4) = 22\tfrac{1}{3}$

e $\int_0^4 \left|t^2 - 9t + 18\right| \; dt = 23\tfrac{2}{3}$

f $v(4) = -2$ so speed is 2 m/s

3 **a** $\int \dfrac{1}{y} dy = \int x^2 \, dx \Rightarrow \ln y = \dfrac{x^3}{3} + c \Rightarrow y = Ae^{\frac{x^3}{3}}$

$y(0) = 3 \Rightarrow A = 3 \quad y = 3e^{\frac{x^3}{3}}$

b $y(2) = 3e^{\frac{8}{3}}$

c using $x_{n+1} = x_n + 0.5 \quad y_{n+1} = y_n + 0.5 y_n x_n^2$

gives $y(2) \simeq 10.758 = 10.8 \, (3 \, sf)$

d $\left|\dfrac{43.176 - 10.758}{43.176}\right| \times 100\% = 75.1\% \, (3 \, sf)$

e Decrease the value of the step length.

4 **a** $\dfrac{dr}{dt} = k4\pi r^2$

b $\int \dfrac{1}{r^2} dr = \int k4\pi dt \Rightarrow \dfrac{-1}{r} = k4\pi t + c$

$t = 0, r = 10 \Rightarrow c = \dfrac{-1}{10}$

$t = 100, r = 5 \Rightarrow \dfrac{-1}{5} = k4\pi 100 - \dfrac{1}{10} \Rightarrow k = \dfrac{-1}{4\pi 1000}$

$\dfrac{-1}{r} = \dfrac{-t}{1000} - \dfrac{1}{10} = -\left(\dfrac{100+t}{1000}\right) \Rightarrow r = \dfrac{1000}{100+t}$

c $t = 900 \Rightarrow r = 1$

d $\frac{1}{10} = \frac{1000}{100+t} \Rightarrow 100+t = 10000 \Rightarrow t = 9900$

e $r \to 0$

11.5 Slope fields and differential equations

1 A function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

a Sketch the slope field for this differential equation for $-2 \le x \le 2, -2 \le y \le 2$.

Isoclines are curves on a slope field for which $\frac{dy}{dx}$ is constant.

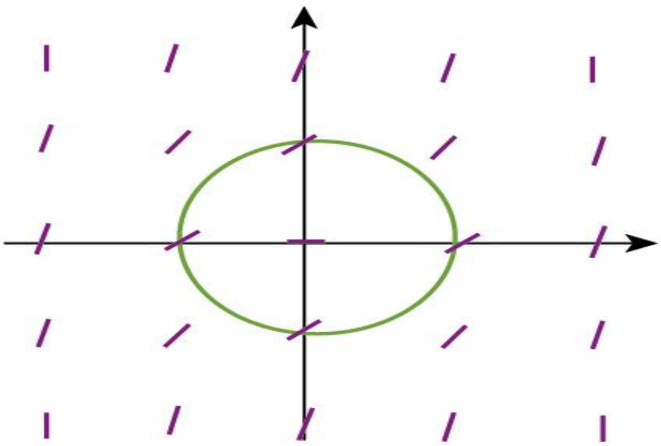
b On your sketch draw in the isocline corresponding to $\frac{dy}{dx} = 1$.

c If the function $y(x)$ were to have a stationary point, state what the co-ordinates of that point must be.

d Explain what type of stationary point it would have to be

Answers

1 **a** **b**



c $\frac{dy}{dx} = x^2 + y^2 = 0 \Rightarrow x = 0, y = 0$ point is (0,0)

d as $\frac{dy}{dx} = x^2 + y^2 > 0$ except at this point, it cannot be a max or a min so must be a horizontal point of inflection.

12.1 Vector quantities

1 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ k \\ -2 \end{pmatrix}$. Given that **a** and **b** are perpendicular, find the value of the constant k.

2 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find the angle between the directions of **a** and **b** giving your answer in degrees.

3 Let $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

a Find a unit vector parallel and in the same direction as **a**.
b Find two unit, 2D vectors perpendicular to **a**.

4 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

a Find the scalar component of **a** in the direction of **b.**
b Find the scalar component of **a** perpendicular to **b.**
c Let your answer to part (a) be x and your answer to part (b) be y.

State what the quantity $\sqrt{x^2 + y^2}$ must be equal to. Check that this is the case.

Answers

1 Taking dot product $3+2k-6=0 \Rightarrow k=\dfrac{3}{2}$

2 Taking dot product $3+8=11=\sqrt{5}\times 5\cos\theta \Rightarrow \cos\theta=\dfrac{11}{5\sqrt{5}} \Rightarrow \theta = 10.3^\circ\ (3\ sf)$

3 $|\mathbf{a}|=5$

a $\begin{pmatrix}\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$ **b** $\pm\begin{pmatrix}\frac{4}{5}\\ \frac{-3}{5}\end{pmatrix}$

4 **a** $\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}3\\1\\-2\end{pmatrix} / \sqrt{9+1+4}=\dfrac{-1}{\sqrt{14}}$

b $\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}\right| / \sqrt{14}=\left|\begin{pmatrix}-7\\11\\-5\end{pmatrix}\right| / \sqrt{14}=\dfrac{\sqrt{195}}{\sqrt{14}}$

c It must equal $|a|$ which is $\sqrt{1+4+9}=\sqrt{14}$

Checking $\sqrt{\dfrac{1}{14}+\dfrac{195}{14}}=\sqrt{\dfrac{196}{14}}=\sqrt{14}$

12.2 Motion with variable velocity

1 The displacement of an object is given by

$$\mathbf{r}(t) = \begin{pmatrix} \frac{t^3}{3} - 3t^2 + 8t + 2 \\ \frac{t^2}{2} - 2t + 1 \\ 4 \end{pmatrix} \qquad \text{where } t \geq 0 \text{ represents time.}$$

a Write down the initial displacement.
b Find the velocity vector.
c Write down the initial velocity.
d Find the initial speed.
e Investigate if and when the object ever comes to rest.
f Find the acceleration vector.
g Find the initial acceleration (as a vector).

2 A stone is thrown out from the top of a vertical cliff. Its position vector (from an origin at the base of the cliff), $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where x is in an easterly direction, y is in an northerly direction and z is vertically upwards, is a function of time. Units are in metres and seconds.

Its acceleration vector is given by $\mathbf{a}(t) = \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix}$.

Its initial velocity is $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and its initial displacement is $\begin{pmatrix} 0 \\ 0 \\ 125 \end{pmatrix}$.

a Find its velocity vector as a function of time.
b Find its position vector as a function of time.
c Find how long it takes before it hits the ground.
d Find its velocity vector at this time.
e Find its speed at this time, giving your answer to 3 significant figures.
f Find its position vector at this time.
g Find the average speed of the stone over its whole journey from the top of the cliff to the bottom, giving your answer to 3 significant figures.

Answers

1 **a** $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

b $\begin{pmatrix} t^2 - 6t + 8 \\ t - 2 \\ 4 \end{pmatrix}$

c $\begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix}$

d $\sqrt{8^2 + 2^2} = \sqrt{68}$

e At rest $\Rightarrow (t-2)(t-4) = 0$ and $t - 2 = 0$. So yes, does come to rest at $t = 2$

f $\begin{pmatrix} 2t - 6 \\ 1 \\ 0 \end{pmatrix}$

g $\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}$

2 **a** Integrating $\begin{pmatrix} 2 \\ 3 \\ -10t \end{pmatrix}$

b Integrating again $\begin{pmatrix} 2t \\ 3t \\ -5t^2 + 125 \end{pmatrix}$

c $-5t^2 + 125 = 0 \Rightarrow t = 5\text{ s}$

d $\begin{pmatrix} 2 \\ 3 \\ -50 \end{pmatrix}$

e $\sqrt{50^2 + 2^2 + 3^2} = \sqrt{2513} = 50.1$ m/s (3 sf)

f $\begin{pmatrix} 10 \\ 15 \\ 0 \end{pmatrix}$

g total distance/time= $\sqrt{10^2 + 15^2 + 125^2}$ / 5 = 25.3 m/s (3 sf)

12.3 Exact solutions of coupled differential equations

1 In a large game reserve there are wildebeest and lions that prey on the wildebeest. Let the number of wildebeest be x measured in thousands, the number of lions be y measured in hundreds and let time t be measured in years. The following paired differential equations are satisfied:

$$\dot{x} = 0.6x - 0.3xy$$
$$\dot{y} = 0.1xy - 8y$$

a Find the equilibrium state with both values as positive.

b Initially let the number of wildebeest be x_0 thousands and the number of lions be y_0 hundreds. In the quarter-plane $x_0 > 0, y_0 > 0$ draw in two appropriate lines and in the interior of each of the 4 regions created, state if the number of

i wildebeest **ii** lions

will be either increasing or decreasing.

2 A Catherine wheel firework is attached to a rocket. The wheel rotates in a vertical plane. The rocket is fired vertically upwards from a small planet with negligible gravity. A point P is on the circumference of the Catherine wheel and is initially directly above the centre of the wheel. The position vector of $P = \begin{pmatrix} x \\ y \end{pmatrix}$ where x is a horizontal distance measured in metres and y is a vertical distance measured in metres. Time t is measured in seconds. The rocket is fired upwards at $t = 0$ and has a constant speed of 10 m/s. The initial position vector for $P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The velocity vector for P is given by $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} w\cos wt \\ 10 - w\sin wt \end{pmatrix}, w > 0$.

a If the point P is momentarily at rest at certain times, find the value of w.

b Find the position vector of P as a function of time.

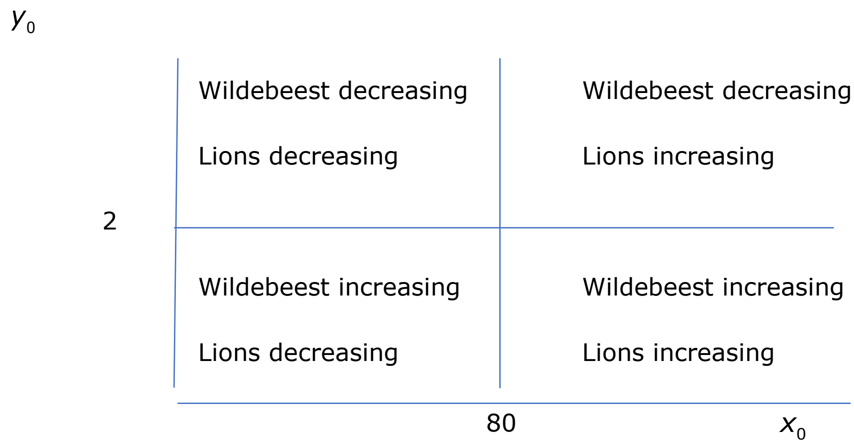
c Find the acceleration vector of P as a function of time.

d Find the radius of the Catherine wheel.

Answers

1 **a** $\dot{x} = 0 \Rightarrow y = 2$ hundred lions $\quad \dot{y} = 0 \Rightarrow x = 80$ thousand wildebeest

b



2 **a** At rest gives

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} w\cos wt \\ 10 - w\sin wt \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \cos wt = 0 \Rightarrow \sin wt = \pm 1 \Rightarrow w = 10$$

b

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin 10t \\ 10t + \cos 10t - 1 \end{pmatrix}$$

c

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -100\sin 10t \\ -100\cos 10t \end{pmatrix}$$

d Radius of the wheel will be given by the largest x value, which is 1 m.

12.4 Approximate solutions to coupled linear equations

1 Let $\mathbf{M} = \begin{pmatrix} 0.2 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}$

a Find the eigenvalues for matrix $\mathbf{M}$.
b For each eigenvalue find the corresponding eigenvectors.

Two types of antelopes, type A and type B, are competing for resources on the plains. Let x measured in thousands represent the number of type A antelopes and let y, measured in thousands, represent the number of type B antelopes, at time t measured in decades.

These variables satisfy the paired differential equations:

$$\begin{aligned} \dot{x} &= \ \ 0.2x - 0.1y \\ \dot{y} &= -0.2x + 0.3y \end{aligned}$$

Initially $x = 8$ and $y = 20$.

c Use parts (a) and (b) to solve these differential equations to find both x and y as explicit functions of t.

d Find the population of type A and type B antelopes after

i one decade **ii** two decades.

e Calculate how many years it will take for the population of type A antelopes to essentially die out and state the number of type B antelopes at this time.

f State, with a reason if the solutions found in part (c) will be applicable after the number of years found in part (e).

Let s be the number of years, in decades, after the number of years found in part (e).

The paired differential equations reduce to one single differential equation.

g State this differential equation and then solve it.

h Hence find the number of type B antelopes when $s = 1$ decade.

2 In the spirit of a Paper 3 investigation question.

Consider the differential equation

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 8x = 0$$

a If $x = e^{\alpha t}$ is to be a solution, find the two values of α that make this possible. Call these values α_1 and α_2.

b Show that $x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$ is a solution for any constants A and B.

Let $y = \frac{dx}{dt}$, then as an alternative method, we have the paired differential equations

$\dot{x} = 0x + y$

$\dot{y} = -8x + 6y$

c Solve these equations by the eigenvalue method and check that you have the same answer as previously.

Now consider the harder differential equation

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 8x - 8t = 0$$

d Find a and b so that $x = at + b$ is a particular solution to this differential equation.

e Combine this particular solution with part (b) to find the general solution.

f Find what this general solution will become given that $x = 2\frac{3}{4}$ and $\frac{dx}{dt} = 7$ when $t = 0$ and hence calculate x when $t = 1$.

g Apply Euler's method with a step length of 0.01 to the paired differential equations

$$\dot{x} = 0x + y$$
$$\dot{y} = -8x + 6y + 8t$$

See if an answer for $x(1)$ is obtained that is close to the solution found in part (f).

Answers

1 **a** $\begin{vmatrix} 0.2-\lambda & -0.1 \\ -0.2 & 0.3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 0.5\lambda + 0.04 = 0 \Rightarrow (\lambda - 0.1)(\lambda - 0.4) = 0$

$\lambda = 0.1$ or 0.4

b $\lambda = 0.1$ $\begin{pmatrix} 0.2 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = y$

Eigenvectors are of the form $\begin{pmatrix} t \\ t \end{pmatrix}$

$\lambda = 0.4$ $\begin{pmatrix} 0.2 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.4\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2x + y = 0$

Eigenvectors are of the form $\begin{pmatrix} -s \\ 2s \end{pmatrix}$

c $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.1t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{0.4t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$A - B = 8, A + 2B = 20 \Rightarrow A = 12, B = 4$

$x = 12e^{0.1t} - 4e^{0.4t}$ $y = 12e^{0.1t} + 8e^{0.4t}$

d **i** Type A 7.29 thousand (3 sf) Type B 25.2 thousand (3 sf)

ii Type A 5.75 thousand (3 sf) Type B 32.5 thousand (3 sf)

e $12e^{0.1t} - 4e^{0.4t} = 0 \Rightarrow 3 = e^{0.3t} \Rightarrow t = \dfrac{\ln 3}{0.3} = 3.66$ decades, 36.6 years (3 sf)

When Type B population is 51.9 thousand (3 sf)

f No, cannot have a negative number of antelopes of any species.

g $\dfrac{dy}{ds} = 0.3y$

$\int \frac{1}{y}\,dy = \int 0.3\,ds \Rightarrow \ln y = 0.3s + c \Rightarrow y = Ae^{0.3s}$

$y = 51.9e^{0.3s}$

h 70.1 thousand (3 sf)

2 **a** $\alpha^2 e^{\alpha t} - 6\alpha e^{\alpha t} + 8e^{\alpha t} = 0 \Rightarrow \alpha^2 - 6\alpha + 8 = 0 \Rightarrow (\alpha - 4)(\alpha - 2) = 0$

$\alpha = 4$ or 2

b $x = Ae^{4t} + Be^{2t}$, $\dfrac{dx}{dt} = 4Ae^{4t} + 2Be^{2t}$, $\dfrac{d^2x}{dt^2} = 16Ae^{4t} + 4Be^{2t}$

$16\,Ae^{4t} + 4Be^{2t} - 6\left(4Ae^{4t} + 2Be^{2t}\right) + 8\left(Ae^{4t} + Be^{2t}\right) = 0$ as required

c $\begin{vmatrix} -\lambda & 1 \\ -8 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$, $\lambda = 4$ or 2

For $\lambda = 4$ $\quad \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 4\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = 4p$, an eigenvector is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

For $\lambda = 2$ $\quad \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 2\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow q = 2p$, an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{4t} + B\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}$, giving the same solutions as before.

d For $x = at + b,\ \dot{x} = a,\ \ddot{x} = 0$ require

$-6a + 8at + 8b - 8t = 0 \Rightarrow a = 1, b = \dfrac{3}{4}$

e Since $x = Ae^{4t} + Be^{2t}$ makes $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 8x = 0$ and $x = t + \dfrac{3}{4}$ makes

$\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 8x - 8t = 0$ general solution is $x = Ae^{4t} + Be^{2t} + t + \dfrac{3}{4}$

f require $A + B + \frac{3}{4} = 2\frac{3}{4}$ and $4A + 2B + 1 = 7 \Rightarrow A = 1, B = 1$

solution is $x = e^{4t} + e^{2t} + t + \dfrac{3}{4}$

$x(1) = e^4 + e^2 + 1 + \dfrac{3}{4}$ = 63.7 (3 sf)

g Using $t(n+1) = t(n) + 0.01$ Let $\dot{x} = y,\ x(n+1) = x(n) + 0.01y(n)$

$\dot{y} = 6y - 8x + 8t\,, y(n+1) = y(n) + 0.01\left(6y(n) - 8x(n) + 8t(n)\right)$

gives $x(1) \simeq 59.5(3\ sf)$

13.1 Modelling random behaviour

1 A biased five-sided die is thrown and the number obtained is B. The probability distribution of B is defined by this table:

b	1	2	5	6	7
$P(B=b)$	$P(B=1)$	0.1	0.5	0.2	0.1

a Find:

i $P(B=1)$ **ii** $P(B \leq 4)$ **iii** $P(B<7)$

iv $P(B$ is less than 4$)$ **v** $P(B$ is more than 5$)$

b Find the minimum number of throws n of this die required so that the probability of throwing no prime number in n trials is less than 0.0001.

2 A discrete probability distribution is defined by the function $P(Y=y)=cy(5-y)$ for y = 1, 2, 3, 4. Find the value of c and of E(Y).

3 Two fair four-sided dice are numbered 3, 1, 4 and 9. The dice are thrown and the numbers added together to define a discrete random variable T.

a Represent the probability distribution of T in a table.

b A game is played in which if T is prime, a prize of \$5.20 is awarded, otherwise the prize is \$2. If the game costs \$X to play, find the value of X that makes the game fair.

4 A box contains 5 uncharged and 2 charged batteries. Batteries are selected one at a time, without replacement, up to and including when the first charged battery is drawn. Let C denote the total number of batteries drawn.

a Find the probability distribution of C **b** Hence find E(C).

5 Shipping containers processed in a port can have capacity 180 or 210 cubic metres only. A container is selected at random and its capacity defines a random variable Y. Given that E(Y) = 200 cubic metres, find the probability distribution table of Y.

6 A discrete random variable U has probability distribution function $P(U=u)=t\left(\frac{3}{7}\right)^u$, $u \in \mathbb{N}$

Find the value of t.

7 Passengers on Calair have three options for an appetizer, costing 4, 5 or 7 euros, and three options for a main course, costing 9, 10 or 15 euros. Assuming that a passenger who selects an appetizer or a main course does so at random, the probability no appetizer is chosen is 0.2 and the probability that no main course is chosen is 0.1, find the probability distribution table of F, the total spent by a randomly chosen passenger, and find E(F) to two decimal places.

Answers

1 **a** **i** $P(B=1)=0.1$ **ii** $P(B \le 4) = 0.2$ **iii** $P(B<7) = 0.9$

iv $P(B$ is less than $4) = 0.2$ **v** $P(B$ is more than $5)=0.3$

b $P(\text{no prime number in n trials}) = 0.3^n$. $0.3^n < 0.0001 \Leftrightarrow n\ln(0.3) < \ln(0.0001)$

hence $n > \dfrac{\ln(0.0001)}{\ln(0.3)} \approx 7.65$ so $n = 8$.

2 $\sum_{y=1}^{4} P(Y=y) = \sum_{y=1}^{4} cy(5-y) = c\sum_{y=1}^{4} y(5-y) = 1 \Rightarrow c = \dfrac{1}{1\times 4+2\times 3+3\times 2+4\times 1} = \dfrac{1}{20}$

$E(Y) = \sum_{y=1}^{4} yP(Y=y) = \sum_{y=1}^{4}\dfrac{1}{20}y^2(5-y) = \dfrac{1}{20}\sum_{y=1}^{4} y^2(5-y) = \dfrac{1}{20}\left(1\times 4+2\times 3+9\times 2+16\times 1\right) = 2.2$

3 **a** Construct a sample space diagram in order to determine the probability distribution table:

t	2	4	5	6	7	8	10	12	13	18
$P(T=t)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b The expected winnings are $5.2\times\dfrac{6}{16} + 2\times\dfrac{10}{16} = 3.2$, so the game should cost \$3.20 to play.

4 **a** Construct a tree diagram in order to determine the probability distribution table

c	1	2	3	4	5	6
$P(C=c)$	$\frac{6}{21}$	$\frac{5}{21}$	$\frac{4}{21}$	$\frac{3}{21}$	$\frac{2}{21}$	$\frac{1}{21}$

b $E(C) = \sum_{c=1}^{6} cP(C=c) = \dfrac{1}{21}\left(1\times 6+2\times 5+3\times 4+4\times 3+5\times 2+6\times 1\right) = \dfrac{8}{3}$.

5 If p is the proportion of shipping containers with capacity 180 cubic metres, then

$180\times p + 210\times(1-p) = 200$, hence $p = \dfrac{1}{3}$ and the probability distribution table is

y	180	210
$P(Y=y)$	$\frac{1}{3}$	$\frac{2}{3}$

6 $\sum_{u=0}^{\infty} P(U = u) = \sum_{u=0}^{\infty} t\left(\frac{3}{7}\right)^u = 1$ hence $t + \sum_{u=1}^{\infty} t\left(\frac{3}{7}\right)^u = t + t\frac{\frac{3}{7}}{1-\frac{3}{7}} = 1$ so $t = \frac{4}{7}$.

7 Construct a tree diagram in order to determine the probability distribution table:

f	2	4	5	6	7	8	10	12	13	18
$P(F=f)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Hence E(F) = €12.95

13.2 Modelling the number of successes in a fixed number of trials

1 Given $D \sim B(8, 0.17)$, find the probabilities:

a $P(D = 4)$ **b** $P(D > 2)$ **c** $P(0 < D < 8)$

d $P(D \text{ is less than or equal to } 0)$.

e The values of the discrete random variable D define a game in which the prize is £D. What should this game cost to play in order for it to be fair?

2 Quality control investigations in a factory determine that 2.87% of the lightbulbs they produce are faulty. Find the probability that in a batch of 400 bulbs, between 2.9% and 4.1% of the bulbs are faulty, stating any assumptions that you make.

3 The English diarist Samuel Pepys wrote to Sir Isaac Newton in 1693 with this question: If player *A* throws six dice and has to throw at least one six, player B has twelve dice and has to throw at least two sixes and player C has eighteen dice and has to throw at least three sixes, do they have the same chance of winning?

a Model the situations and find the probabilities needed to answer Samuel's question.

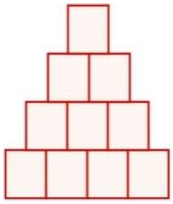
b Make a general statement for the probability of throwing at least n sixes with $6n$ dice, using sigma notation.

4 The random variable H follows a binomial distribution with mean 3.36 and variance 2.5536 exactly. Find the probability that H is 4 or more.

5 3.1% of the potatoes in a harvest have an irregular shape. Find the largest number of potatoes that can be chosen in a sample so that the probability that there are no irregular potatoes is at least 0.5.

6 A botanical garden plants giant sunflower seedlings in 15 rows of 10 seedlings. Assuming that each seedling survives independently of each other and grows into a giant sunflower with probability 0.7, find the probability that at least one row has at least 9 giant sunflowers.

7



Isabella throws a ball at a tower of ten cans in a fairground game where the object is to knock over the tower. Isabella knows from experience that she needs to hit at least three cans in order to knock over the tower. The probability that she hits any can is 0.2 and her throws are independent of each other. Find the minimum number of balls she must throw at the tower for the probability of knocking down the tower to be at least 0.95.

Answers

1 **a** $P(D=4)$ = 0.0277

b $P(D>2)=$ $P(D\geq 3)$ = 0.141

c $P(0<D<8)=P(1\leq D\leq 7)=0.775$

d $P(D$ is less than or equal to 0$)=P(D=0)=0.225$

e The expected winnings is the same as E(D) which is $8\times 0.17=1.36$, so the game should cost £1.36 in order to be fair.

2 2.9% and 4.1% of 400 lightbulbs represent 11.6 and 16.4 lightbulbs. The integers between these numbers are 12 and 16. If 2.87% of the batch of 400 bulbs are faulty and if the 400 bulbs were selected randomly, then the number of faulty bulbs F can be modelled by a binomial distribution $F \sim$ B(400, 0.0287). The required probability is $P(12\leq F\leq 16)=0.406$.

3 **a** If A, B and C are respectively the number of successes in each experiment, then $A \sim$ B(6, 1/6), $B \sim$ B(12, 1/6), $C \sim$ B(18, 1/6), $P(A\geq 1)$, $P(B\geq 2)$ and $P(C\geq 3)$ are the distributions and events needed to answer Samuel's question. These probabilities are 0.665, 0.619 and 0.597 respectively. Hence the answer to Samuel's question is "No".

b P(Throw at least n sixes with 6n dice)= $\sum_{i=n}^{6n} {}^{6n}C_i \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{6n-i}$

4 H ~ B(n,p).and E(H) = 3.36 hence np = 3.36. Var(H) = 2.5536 = np(1-p) hence (1-p) = $\frac{2.5536}{3.36}$ = 0.76, therefore p = 0.24 and n = $\frac{3.36}{0.24}=14$. $P(H\geq 4)=0.443$.

5 3.1% of the potatoes in a harvest have an irregular shape. Find the largest number of potatoes that can be chosen in a sample so that the probability that there are no irregular potatoes is at least 0.5.

If I is the number of irregular potatoes in a sample of n, then $I \sim$ B(n, 0.031)
$P(I=0)=(1-0.031)^n=0.969^n$.

$0.969^n>0.5 \Rightarrow n\ln(0.969)>\ln(0.5) \Rightarrow n<\frac{\ln(0.5)}{\ln(0.969)}=22.0122$. Hence 22 is the answer required.

6 In each row of 10 seedlings, the number of seedlings S that grow into a giant sunflower follows a binomial distribution $S \sim$ B(10, 0.7), so the probability that a row has at least nine giant sunflowers is p = $P(S\geq 9)=0.149308\ldots$ The number of rows that have at least nine giant sunflowers follows a binomial distribution R ~ B(15, p)

The probability required is $P(R\geq 1)=0.912$

7 If H represents the number of balls that hit the tower, then $H \sim$ B(n, 0.2)

$P(H\geq 3)\geq 0.95 \Rightarrow P(H\leq 2)\leq 0.05$ can be solved with technology to find that n = 30.

13.3 Modelling the number of successes in a fixed interval

1 Use technology to find the following probabilities given that $B \sim \text{Po}(4.7)$.

a $P(B = 5)$ **b** $P(B \leq 5)$ **c** $P(B$ is at least $2)$

d $P(B$ is no more than $E(B))$

2 The number of microscopic flaws *M* per one square metre of a sheet of aluminium follows a Poisson distribution with variance 0.057. Find the following probabilities:

a $P(Y = 0)$

b In a panel measuring 30 square metres, there are at least 5 flaws.

If in a batch of 50 such panels, at least 4 have at least five flaws, then the batch fails quality control and is rejected.

c Find the probability that in a consignment of 100 batches, more than six are rejected.

3 Xinyu manages tourist boat trips to view whales. She finds that the number of whale sightings from her boat per hour follows a Poisson distribution with mean 2.1.

a Find the probability that the tourists on Xinyu's boat:

i see at least one whale in the first hour of a boat trip

ii see at least 5 whales if the trip lasts 2 hours 45 minutes.

Xinyu defines a boat trip in which her clients see more than 6 whales as *optimal.*

b Find the least number of minutes *m* that a boat trip needs to be so that the probability of the trip being optimal is at least 90%.

c With this value of *m*, what is the probability that at least 40 trips from a season of 50 trips are optimal? State any assumptions that you make.

Answers

1 **a** $P(B=5)=0.174$ **b** $P(B\le 5)=0.668$ **c** $P(B \text{ is at least } 2)=0.948$

d $P(B \text{ is no more than } E(B)) = P(B\le 4.7) = P(B\le 4) = 0.495$.

2 **a** $P(Y=0) = 0.945$

b In this panel, the number of flaws F follows a Poisson distribution with parameter $30\times 0.057 = 1.71$. If $F \sim \text{Po}(1.71)$ then $P(F\ge 5) = 0.030255.... = q$

c The number of panels P in a batch of 50 that have at least five flaws follows a binomial distribution $P \sim \text{B}(50, q)$. $P(P\ge 4) = 0.064332... = r$. The number of failing batches in a consignment of 100 B follows a binomial distribution $B \sim \text{B}(100, r)$, hence $P(B\ge 7) = 0.465$

3 **a** **i** If $W \sim \text{Po}(2.1)$ then $P(W\ge 1) = 0.878$

ii $T \sim \text{Po}(2.1 \text{ x } 2.75)$ then $P(T\ge 5) = 0.684$.

b Let $a = \dfrac{2.1m}{60}$ represent the expected number of whale sightings per m minutes.

We require that if $W_0 \sim \text{Po}(a)$, then $P(W_o \ge 7)\ge 0.9$. Use your GDC to find the intersection between the graphs y = 0.9 and the cumulative Poisson probability function with parameter x such that y= $P(W_o\ge 7)$. You will find that the solution to $P(W_o\ge 7) = 0.9$ is x = 10.5321, giving $10.5321 = \dfrac{2.1m}{60}$, giving m =301 minutes.

c Using m = 301 minutes gives the probability of a trip being optimal as 0.9. Assuming that the probability of an optimal trip is constant over the entire season, the number of optimal trips O follows a binomial distribution $O \sim \text{B}(50, 0.9)$. The required probability is $P(O\ge 40)$ =0.991.

13.4 Modelling measurements that are distributed randomly

1 $T \sim N(6, 1)$

a Write down these probabilities:

i $P(T > 7)$ **ii** $P(T < 4)$

b Use technology to find these probabilities:

i $P(T \le 6.21)$ **ii** $P(T > 5.79)$

iii $P(T \le 6.05 | T > 5.1)$ **iv** $P(T > 7 | T < 8)$ **v** $P(T < 5 | T > 4)$

2 Valentina collects data over many months about the length of her daily bus journey and finds that the time taken for her journey T is normally distributed with mean 35.4 minutes and standard deviation 5.7 minutes. Valentina resolves to complain to the bus company every time the bus journey takes more than 40 minutes. Calculate the number of days on which she would expect to make a complaint in a working week of 5 days. State any assumptions you make.

3 Environmental surveys showed that weights of salmon in a Scottish river are normally distributed with a mean of 2.1 kg and a standard deviation of 450 g.

a Calculate the probability that a salmon that is caught weighs more than 2.3 kg

b Erin catches 7 salmon in one day. Calculate the probability that more than 4 of them weigh more than 2.3 kg.

c Calculate the probability that a salmon caught weighs more than 3 kg, given that it weighs more than 2.5 kg.

4 A large sample of satsumas is surveyed and their radii R are measured. R is found to follow a normal distribution with mean 2.7 cm and standard deviation 0.6 cm. Satsumas with a radius 1.8 cm or less are classified as small, whereas a satsuma with a radius 3.5 cm or more is classified as large. Two satsumas are chosen at random from the population without replacement. What is the probability that one large and one small satsuma are chosen?

5 Two solar panels A and B have lifespans that are normally distributed as follows:

	Mean (hours)	Standard deviation
A	9127	710
B	10018	997

a Panel A is guaranteed to last at least 7500 hours. Find the probability that a randomly selected panel A will fail before the end of the guarantee.

b Panel B is advertised to last at least t hours with probability 99.9%. Calculate the value of t correct to the nearest 100 hours.

An engineering company explores two design scenarios for a solar powered remote weather station. Scenario I involves a design using eight A panels in which the weather station will function if at least 3 of the panels are working. Scenario II uses five B panels and will fail to function if at least one panel fails. Solar panels operate independently of each other.

c Find the probabilities under each scenario that the weather station is functioning after one year of continuous use.

Answers

1 **a** **i** $\dfrac{1-0.68}{2} = 0.16$ **ii** $\dfrac{1-0.95}{2} = 0.025$

b **i** 0.583 **ii** 0.583

iii $P(T \le 6.05 | T > 5.1) = \dfrac{P(5.1 < T < 6.05)}{P(T > 5.1)} = 0.412$

iv 0.139 **v** 0.139

2 $T \sim \mathrm{N}(35.4, 5.7^2)$. $P(T > 40) = 0.209828\ldots = p$. Assuming that the probability of the journey taking more than 40 minutes has the constant value of p for each of the five days, Valentina can work out the expected number of days by finding $5p = 1.05$ days.

3 **a** Let W be the weight of a randomly selected salmon from the river.

Then $W \sim \mathrm{N}(2.1, 0.45^2)$. $P(W > 2.3) = 0.328361\ldots = q$

b If C represents the number of the 7 salmon caught that weigh more than 2.3 kg, then $C \sim \mathrm{B}(7, q)$. $P(C \ge 5) = 0.0425$

c $P(W > 3 | W > 2.5) = \dfrac{P(W > 3)}{P(W > 2.5)} = 0.122$.

4 $R \sim \mathrm{N}(2.7, 0.6^2)$. The probability that a randomly selected satsuma is small is therefore $P(R < 1.8) = 0.066807\ldots = r$ and the probability that a randomly selected satsuma is large is $P(R > 3.5) = 0.091211\ldots = s$. The probability required can be found from a tree diagram or from $2rs$ directly; $2rs = 0.0122$.

5 **a** If L_A represents the lifespan of a randomly selected solar panel A, then $L_A \sim \mathrm{N}(9127, 710^2)$ and $P(L_A \le 7500) = 0.0110$

b If L_B represents the lifespan of a randomly selected solar panel B, then $L_B \sim \mathrm{N}(10018, 997^2)$ and $P(L_B \ge t) = 0.999$. You can use the inverse cumulative normal function to solve $P(L_B < t) = 0.001$, giving t = 6900 hours to the nearest 100 hours.

c The probability that a panel A is working after one year is $P(L_A \ge 8760) = 0.697388\ldots = p$. The panels operate independently of each other so the number of the 8 panels still working after one year X follows a binomial distribution: $X \sim \mathrm{N}(8, p)$. $P(X \ge 3) = 0.988$.

The probability that a panel B is working after one year is $P(L_B \ge 8760) = 0.896487\ldots = q$. All five panels must be working under scenario II: $q^5 = 0.579$.

Hence the probability that a weather station designed with scenario I will be functioning after one year is 0.988 and the probability that one designed with scenario II will still be functioning after one year is 0.579.

13.5 Mean and variance of transformed or combined random variables

1 A, B and C are independent random variables such that $A \sim B(10, 0.2)$, $B \sim Po(6.1)$ and $C \sim N(7, 1.1^2)$. Calculate:

a $E(3A + 8)$ **b** $Var(3A - 6)$ **c** $E(A + B - 2C)$ **d** $Var(4A - 3B + C)$

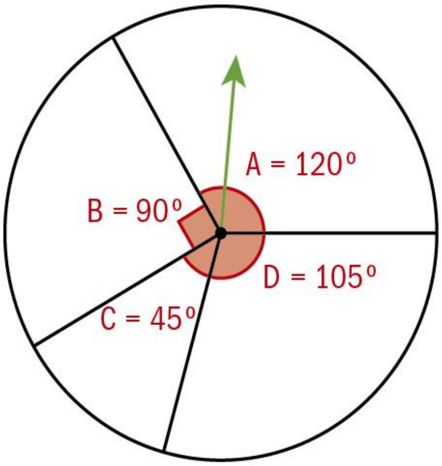
2 The random variable Y is such that $E(3Y - 2) = 4$ and $Var(4 - Y) = 9$.

Calculate **a** $E(Y)$ **b** $Var(Y)$

3 The independent random variables X and Y have standard deviations $\sqrt{2}$ and $\sqrt{3}$ respectively. Find p and q, where $p, q \in \mathbb{R}^+$ such that $Var(5pX + qY) = 10$ and $Var(pX2qY) = 20$.

4

Xinyu and Erin are exploring a game of chance. Xinyu designs the spinner shown to the right. The arrow rotates and stops in sector A, B, C or D at random, scoring 5, 4, 1 or 3 points respectively. This defines the random variable X. Erin designs a cubical dice numbered 1, 1, 1, 3, 5, 5 and the number thrown on her dice gives the number of points scored, defining the random variable R.



a Find $E(X)$ and $E(R)$.

b Determine the probability distribution table of $X+R$

c Hence confirm that $E(X + R) = E(X) + E(R)$.

Answers

1 **a** $E(3A+8) = 3E(A) + 8 = 3\times10\times0.2+8=14$

b $\text{Var}(3A-6) = 9\text{Var}(A) = 9\times10\times0.2\times0.8=14.4$

c $E(A+B-2C) = E(A)+E(B)-2E(C) = 10\times0.2+6.1-2\times7=-5.9$

d $\text{Var}(4A-3B+C) = 16\text{Var}(A)+9\text{Var}(B)+\text{Var}(C) =$
$16\times10\times0.2\times0.8+9\times6.1+1.1^2=81.71$

2 **a** $E(3Y-2)=4\Rightarrow 3E(Y)-2=4\Rightarrow E(Y)=2$

b $\text{Var}(4-Y)=9\Rightarrow \text{Var}(Y)=9$

3 $\text{Var}(5pX+qY)=25p^2\text{Var}(X)+q^2\text{Var}(Y)=50p^2+3q^2=10$

$\text{Var}(pX-2qY)=p^2\text{Var}(X)+4q^2\text{Var}(Y)=2p^2+12q^2=20$

Using technology to solve the system of simultaneous equations, $p^2=\frac{10}{99}\Rightarrow p=\frac{1}{3}\sqrt{\frac{10}{11}}$ and

$q^2=\frac{490}{297}\Rightarrow q=\frac{7}{3}\sqrt{\frac{10}{33}}$.

4 The probability table for X is

sector	A	B	C	D
x	5	4	1	3
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{7}{24}$

Hence $E(X)=5\times\frac{1}{3}+4\times\frac{1}{4}+1\times\frac{1}{8}+3\times\frac{7}{24}=\frac{11}{3}$

The probability table for R is

number	1	3	5
r	1	3	5
$P(R=r)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$

Hence $E(R)=1\times\frac{1}{2}+3\times\frac{1}{6}+5\times\frac{1}{3}=\frac{8}{3}$

The probability table for $X+R$ is found by drawing a tree diagram:

$x+r$	2	4	5	6	7	8	9	10
$P(X+R)= x+r$	$\frac{1}{16}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{37}{144}$	$\frac{1}{24}$	$\frac{11}{72}$	$\frac{1}{12}$	$\frac{1}{9}$

Hence $E(X+R)=2\times\frac{1}{16}+4\times\frac{1}{6}+5\times\frac{1}{8}+6\times\frac{37}{144}+7\times\frac{1}{24}+8\times\frac{11}{72}+9\times\frac{1}{12}+10\times\frac{1}{9}=\frac{19}{3}$

$E(X)+E(R)=\frac{11}{3}+\frac{8}{3}=\frac{19}{3}$, confirming the result $E(X+R)=E(X)+E(R)$.

13.6 Distribution of combined random variables

1 C and D are independent normally distributed variables such that $C \sim \mathrm{N}(6, 1.3^2)$ and $D \sim \mathrm{N}(4.7, 0.9^2)$. The random variable F is defined by $F = 4C - 2D + 1$.

Find: **a** $\mathrm{E}(F)$ **b** $\mathrm{Var}(F)$ **c** $P(10 \leq F < 13)$.

2 A grocery sells potatoes in three sizes. Weights of *premium* potatoes are distributed normally with a mean of 180 g and a standard deviation of 25 g, weights of *regular* potatoes are distributed normally with mean 150 g and standard deviation 15 g. Weights of *baby* potatoes are distributed normally with mean 40 g and standard deviation 5 g.

a One potato of each type is chosen at random. Find the probability that the regular potato weighs more than the premium potato.

b Find the probability that a randomly chosen premium potato weighs more than 3 times that of a randomly chosen baby potato.

c Find the probability that in a shopping basket of three premium, five regular and ten baby potatoes, the total weight is greater than 1.8 kg.

3 Assume that the mean duration of a human pregnancy is 270 days, and that the standard deviation is 15 days. A maternity ward of a hospital has beds for 32 mothers. Assuming all the beds are occupied, find the probability that the mean length of the 32 pregnancies is

a less than the 259 days that defines a premature birth

b at least 7 days overdue.

4 Managers record the number of adults and the number of children who arrive at a fast food outlet between 17:00 and 19:30. It is found that the number of adults arriving follows a Poisson distribution with mean 9 per 5-minute interval, and the number of teenagers who arrive in a 5-minute interval follows a Poisson distribution with mean 15. Assuming that the adults and children arrive independently of each other, find the minimum number n such that the probability of there being more than n people arriving at the fast food outlet in a 5-minute interval is less than 5%.

5 An apple tree in Callum's garden has a large harvest every year with apples that have an average weight of 107 g and standard deviation 14 g. Callum wants to sell the apples in bags at a local fete and he wants to label the bags "average weight 105 g". What is the minimum number of apples he should include in a bag so that the probability is less than 0.1 that the average weight of the apples in the bag is less than 105 g?

6 A game is designed for a charity fundraiser. A computer simulates flipping six fair coins and throwing five fair cubical dice numbered 1, 1, 2, 3, 3, 5. The score S is the sum of the number of heads plus the number of prime numbers. The player can carry out the experiment 40, 80 or 160 times and the prizes are as follows:

Number of trials (n)	If the mean of n trials is greater than:	Win this Prize
40	6.5	\$5
80	6.5	\$10
160	6.5	\$20

Calculate the expected winnings for each number of trials. If the number of trials is selected at random, what is the least cost of the game to the nearest \$ in order for the game to make a profit for charity?

Answers

1 **a** $E(F) = 4\times 6 - 2\times 4.7 + 1 = 15.6$ **b** $Var(F) = 16\times 1.3^2 + 4\times 0.9^2 = 30.28$

c $P(10 \le F < 13) = 0.164$

2 If P, R and B represent the weights of a randomly chosen *premium*, *regular* or *baby* potato respectively, then $P \sim N(180, 25^2)$, $R \sim N(150, 15^2)$, $B \sim N(40, 5^2)$.

a $P(R > P) = P(R - P > 0)$. $R - P \sim N(-30, 25^2 + 15^2)$. Hence $P(R - P > 0) = 0.486$

b $P(P > 3B) = P(P - 3B > 0)$. $P - 3B \sim N(60, 25^2 + 9\times 5^2)$. Hence $P(P - 3B > 0) = 0.528$

c Let $T = P_1 + P_2 + P_3 + R_1 + R_2 + .. + R_5 + B_1 + B_2 + ..B_{10}$ represent the total weight of the shopping basket. Then $T \sim N(3\times 180 + 5\times 150 + 10\times 40, 3\times 25^2 + 5\times 15^2 + 10\times 5^2)$ giving $T \sim N(1690, 3250)$. The probability required is $P(T > 1800) = 0.0268$

3 Since 32>30, the mean duration $\bar{D}$ of the 32 pregnancies can be modelled with the central limit theorem: $\bar{D} \sim N(270, 15^2/32)$.

a $P(\bar{D} < 259) = 0.0000168$ **b** $P(\bar{D} \ge 277) = 0.00415$.

4 The adults and children arrive independently of each other so the number of people Y follows a Poisson distribution with parameter 24. $Y \sim Po(24)$. $P(Y > n) < 0.05 \Rightarrow P(Y \le n) > 0.95$. Use technology to find the solution set is $n \ge 32$. Hence the solution to the problem is $n = 32$.

5 Assuming that the number of apples included in a bag is greater than 30, the average weight $\bar{A}$ of the n apples can be modelled with the central limit theorem: $\bar{A} \sim N(107, 14^2/n)$ $N\left(107, \frac{14^2}{n}\right)$. The solution of the inequality $P(\bar{A} < 105) < 0.1$ can be found with technology as $n > 80.5$, hence if Callum puts 81 randomly chosen apples in each bag, the probability that the average weight of the apples is less than 105 is less than 0.1.

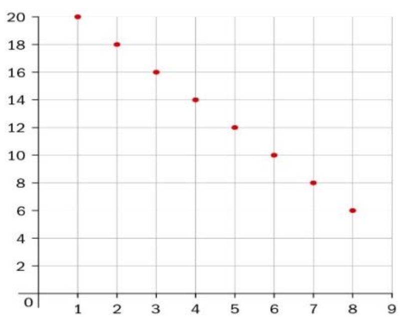
6 If X represents the number of heads in six flips of a fair coin then $X \sim B(6, 0.5)$. Hence $E(X) = 3$ and $Var(X) = 1.5$. If Y represents the number of primes in the five dice throws then $Y \sim B(5, 2/3)$. Hence $E(X) = \frac{10}{3}$ and $Var(X) = \frac{10}{9}$. Therefore the mean of S is $\frac{10}{3} + 3 = \frac{19}{3}$ and the variance of S is $\frac{3}{2} + \frac{10}{9} = \frac{47}{18}$. Since the numbers of trials in the game are all greater than 30, the central limit theorem can be applied. $\bar{S} \sim N(19/3, 47/18n)$. Hence $P(\bar{S} > 6.5) = 0.2570947... = a$, $P(\bar{S} > 6.5) = 0.1781257... = b$ and $P(\bar{S} > 6.5) = 0.0960050... = c$ when n = 40, 80, 160 respectively. The expected winnings are therefore 5a + 10b + 20c = \$4.99. Hence the game will make a profit for charity if the cost of the game is \$5 or more.

14.1 Spearman's rank correlation coefficient

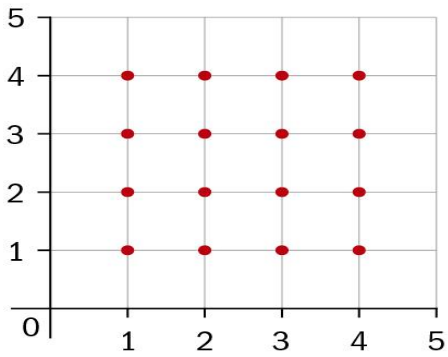
1 **i** For each of the three scatter diagrams given below, write down without any calculation, the value of the Spearman rank correlation coefficient, giving reasons.

ii For each diagram, state if the Pearson product moment correlation coefficient would differ, or not, from the Spearman rank correlation coefficient, giving reasons.

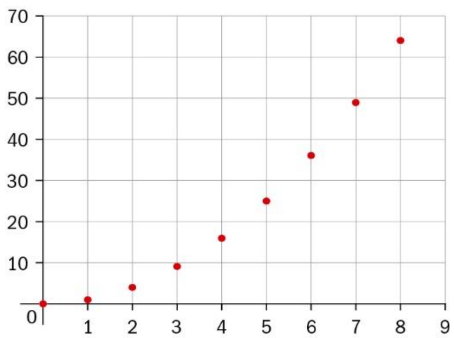
a



b



c



2 The table below gives the positions that 10 football clubs achieved in their league and also their goal difference.

Team	A	B	C	D	E	F	G	H	I	J
Position	1	2	3	4	5	6	7	8	9	10
Goal difference	20	17	15	10	1	3	-1	-5	0	-60

a Create a similar table but replace goal difference with the team's rank based on goal difference, so team A with goal difference of 20 will rank as 1st.

b Find Spearman's rank correlation coefficient for the league position rank against the goal difference rank.

c Comment on the value of r_s obtained in part (b).

d Find the Pearson product moment correlation coefficient for league position against actual goal difference.

e Comment on the value of r obtained in part (d).

f For the actual goal differences use the calculator to find the interquartile range and hence identify any outliers.

g Comment on any difference between the two correlation coefficients obtained and why this might have occurred.

Answers

1 **i** **a** $r_s = -1$ perfect decreasing, monotonic relationship

b $r_s = 0$ no monotonic relationship whatsoever

c $r_s = 1$ perfect increasing, monotonic relationship

ii **a** $r = -1$ perfect negative linear correlation, same

b $r = 0$ no linear correlation whatsoever, same

c $0 < r < r_s$ the relationship is not linear, differs

2 **a**

Team	A	B	C	D	E	F	G	H	I	J
Position	1	2	3	4	5	6	7	8	9	10
G-D rank	1	2	3	4	6	5	8	9	7	10

b $r_s = 0.952$ (3 *sf*)

c this indicates a very strong, increasing, monotonic relationship.

d $r = -0.783$ (3 *sf*)

e this indicates a negative linear correlation (but not incredibly strong).

f $Q_1 = -1$, $Q_3 = 15 \Rightarrow IQR = 16$

$-1 - 1.5 \times 16 = -25$ so poor team J is an outlier.

g $|r_s|$ is closer to 1 than $|r|$ due to the effect of the outlier

Spearman's is less sensitive to outliers.

Weekday	Mon	Tue	Wed	Thur	Fri
Expected absences	20	20	20	20	20

14.2 Hypothesis testing for the binomial probability, the Poisson mean and the product moment correlation coefficient

1 A dog breeder claims that the mean life expectancy of one of her Labrador dogs is 12 years. It is known that the standard deviation of a Labrador dog's life is 2 years. Let $\bar{X}$ be the mean life expectancy of 100 Labradors.

a State the distribution that is approximately satisfied by $\bar{X}$. Write down the name of the theorem that allows you to state this and briefly explain why it applies.

Records show that for a sample of 100 of her dogs, $\bar{x} = 11.8$. We wonder if the breeder is overestimating the mean life expectancy.

b Devise a suitable test to see if the breeder is overestimating. Carry out the test at the 5% level. State the Null and Alternative hypotheses, together with the distribution that would apply under the Null hypothesis. State how many tails the test has. State the *p*-value and the conclusion of the test, with a reason.

c Put the conclusion of the test into the context of the situation.

2 The publisher of a new Maths text book believes that the number of mistakes per page has a mean of 0.2. The publisher will be unhappy if it is more than this. It is known that the number of mistakes per page follows a Poisson distribution. A proofreader looks at 50 pages of the book and finds a total of 12 mistakes.

a Devise a suitable test to see if the publisher's mean can be accepted or if they will be unhappy. Carry out the test at the 10% level. State the Null and Alternative hypotheses, together with the distribution that would apply under the Null hypothesis. State how many tails the test has. State the *p*-value and the conclusion of the test, with a reason.

b Put the conclusion of the test into the context of the situation.

3 The following data shows the number of call-outs an air ambulance service has each day over a period of 200 days.

Number of callouts	0	1	2	3	4	5	≥6
Observed frequency	10	28	52	50	32	22	6

We are going to test at the 5% level if this data could have come from a Poisson distribution with a mean of 3.

a State the name of the test that will be used.

b Give a similar table of expected frequencies and comment on their values in terms of validity of the test.

c State the Null and Alternative hypotheses, the number of degrees of freedom, the *p*-value and the conclusion of the test, with a reason and put the conclusion of the test into the context of the situation.

4 The following data is collected from a random sample of eight fish.

Fish	1	2	3	4	5	6	7	8
Weight (g)	545	554	548	551	558	541	543	549
Length (mm)	351	365	355	353	357	349	348	354

It can be assumed that both the weights and the lengths of the fish come from normal distributions.

a Find the Pearson product moment correlation coefficient r and comment on its value.

This sample is going to be used to test at the 5% significance level whether or not there is an association between the weights and the lengths of the fish. Hypotheses are as follows:

$H_0 : \rho = 0, H_1 : \rho \neq 0$

b For this test find the p-value and hence, with a reason, state the conclusion of the test in context.

Answers

1 **a** $\bar{X} \sim \mathrm{N}\left(12, \frac{2^2}{100}\right)$ by central limit theorem, since n is large

b H_0 : $\mu = 12$

H_1 : $\mu < 12$ so one-tailed (same as the dog!).

Under H_0 , $\bar{X} \sim \mathrm{N}\left(12, \frac{2^2}{100}\right)$

$P(\bar{X} < 11.8) = 0.159 (3\ sf)$

$0.159 > 0.05$ so we accept H_0

c We have no reason to believe that breeder is overestimating the mean.

2 a H_0 : $\mu = 0.2$ for one page

H_1 : $\mu > 0.2$ so one-tailed test.

Let X be the number of mistakes in 50 pages. $\mu = 10$ for 50 pages

Under H_0, $X \sim \mathrm{Po}(10)$

$P(X \geq 12) = 1 - P(X \leq 11) = 0.303\ (3\ sf)$

$0.303 > 0.10$ so we accept H_0

b The publishers do not need to be unhappy and can continue to believe that $\mu = 0.2$.

3 **a** ψ^2 goodness of fit test

b

Number of callouts	0	1	2	3	4	5	≥ 6
Expected frequency	9.96	29.9	44.8	44.8	33.6	20.2	16.8

All the expected values are greater than 5, so there is not a validity problem

c H_0 : this data comes from the $\mathrm{Po}(3)$ distribution

H_1 : this data does not come from the $\mathrm{Po}(3)$ distribution

Degrees of freedom = 6 $p = 0.170\ (3\ sf)$

$0.170 > 0.05$ so we accept H_0 , this data comes from the $\mathrm{Po}(3)$ distribution

4 **a** $r = 0.782\ (3\ sf)$ which suggests a positive association between weight and length.

b p-value $= 0.0219\ (3\ sf) < 0.05$ so we reject H_0 and conclude that there is an association between weight and length

14.3 Testing for the mean of a normal distribution

1 Internet companies claim that the probability that a person will buy a particular object online (rather than go out to shop) is 0.7. To test this claim 20 people who bought this object were interviewed and it was discovered that only 10 had bought online.

a Devise a suitable test to see if the internet companies are overestimating the probability of 0.7. Carry out the test at the 5% level. State the Null and Alternative hypotheses, together with the distribution that would apply under the Null hypothesis.

b State how many tails the test has. State the p-value and the conclusion of the test, with a reason.

c Put the conclusion of the test into the context of the situation.

2 Dougal claims that the mean height of Scottish policemen is greater than the mean height of Welsh policemen. To test this claim the heights of 7 Scottish policemen and 5 Welsh policemen were measured, in centimetres. The results are below:

Scottish: 180, 182, 176, 178, 174, 173, 181

Welsh: 175, 176, 169, 170, 168

It can be assumed that the variance is the same for the heights of Scottish policemen as for the heights of the Welsh policemen.

a State the name of the test that will be used, giving a reason.

b Carry out the test at the 5% level. State the Null and Alternative hypotheses. State how many tails the test has. State the p-value and the conclusion of the test, with a reason.

c Put the conclusion of the test into the context of the situation.

3 The number of hours, X, of effective pain relief given by a particular brand of painkiller, is known to be normally distributed with a mean of μ hours and a standard deviation of 2 hours. A trial was conducted with 20 patients and for this random sample, $\bar{x}$ was found to be 22 hours.

a Find a 90% confidence interval for μ.

b Find a 95% confidence interval for μ.

c The manufacturer of these painkillers claims that they give effective relief for 24 hours. State if your results in parts (a) and (b) support this claim.

d If it was desired that the total length of the confidence interval was to be decreased, state two alterations that could be made to the test in order to achieve this.

4 It is claimed that model A calculator is faster than model B calculator. To test this claim 6 students were asked to do the same calculation on a model A calculator and on a model B calculator. The results in seconds are given in the following table.

Calculator\student	1	2	3	4	5	6
Model A	20	25	15	30	14	34
Model B	22	28	15	35	16	36

a Devise a suitable test to see if the claim can be accepted. State the name of the test that you are using, with a reason. Carry out the test at the 5% level. State the Null and Alternative hypotheses, together with the distribution and any associated parameters that would apply under the Null hypothesis. State how many tails the test has. State the p -value and the conclusion of the test, with a reason.

b Also put the conclusion of the test into the context of the situation.

Answers

1 **a** $H_0: p = 0.7$

$H_1: p < 0.7$

b One-tailed test.

Let X be the number of people who bought online.

Under H_0, $X \sim \mathrm{B}(20, 0.7)$

$P(X \leq 10) = 0.0480 (3\ sf)$

$0.0480 < 0.05$ so we reject H_0

c The companies are overestimating the probability.

2 **a** 2 sample t-test , since variance is unknown

b, c $H_0: \mu_S = \mu_W$ $H_1: \mu_S > \mu_W$ 1-tailed test

p-value = 0.00746 < 0.05 so we reject the Null hypothesis and conclude that Scottish policemen are taller than Welsh policemen

3 **a** $[21.3, 22.7]$ $(3\ sf)$

b $[21.1, 22.9]$ $(3\ sf)$

c The claim is not supported as 24 does not lie in either of the intervals

d Decrease the value of the confidence level

Increase the number of patients taken in the sample

4 **a** This will be a paired sample t-test since the same student uses both calculators and the standard deviation is unknown.

Letting the differences be d we have $d = 2, 3, 0, 5, 2, 2$

$H_0: \mu_d = 0$, $H_1: \mu_d > 0$ t-distribution with 5 degrees of freedom.

1-tailed test

p-value = 0.00864 < 0.05 so we reject the Null hypothesis and conclude that $\mu_d > 0$

b Thus the claim is justified, model A calculator is faster.

14.4 χ^2 test for independence

1 A survey asked people of different ages about their main form of transport as they commuted to work. It is wondered if the age of a person and their form of transport are dependent. The results are given in the table below:

Transport/Age	18-29	30-49	50+
Walk or bike	26	15	5
Car	20	35	22
Train or bus	8	21	32

Carry out a χ^2 test for independence. Do the test at the 5% level. Give a similar table of expected frequencies and comment on their values in terms of validity of the test. State the Null and Alternative hypotheses, the number of degrees of freedom, the p -value and the conclusion of the test, with a reason. Put the conclusion of the test into the context of the situation, with reference to the p -value.

2 Ten random students from a very large school were timed when running 400 m. The results are given in the table below.

Student	1	2	3	4	5	6	7	8	9	10
Time in seconds	70	65	82	59	90	101	100	84	75	120

It can be assumed that these values come from a normal distribution with unknown variance.

a Find the 90% confidence interval for μ, the population mean.

b Find the 95% confidence interval for μ.

c In parts (a) and (b) state, with a reason, the distribution used and any associated parameter.

d If 50 independent samples of 10 students were taken and the 90% confidence interval for μ, the population mean, was calculated in each case , find the expected number of these intervals that would contain μ.

Answers

1 Expected frequencies

Transport/Age	18-29	30-49	50+
Walk or bike	13.5	14.8	14.8
Car	22.6	24.7	24.7
Train or bus	17.9	19.6	19.6

All the expected values are greater than 5, so there is not a validity problem

H_0 : Age and mode of transport are independent

H_1 : Age and mode of transport are dependent

Degrees of freedom = 4 $p = 8.83 \times 10^{-7}$ (3 *sf*)

$8.83 \times 10^{-7} < 0.05$ so we reject H_0

The p-value is so small that we are certain that a person's mode of transport does depend on their age.

2 **a** $[73.8, 95.4]$ (3 *sf*)

b $[71.2, 98.0]$ (3 *sf*)

c t-distribution with 9 degrees of freedom since variance is unknown

d 90% of 50 is 45.

14.5 χ^2 goodness-of-fit test

1 An employer is suspicious that his employees might be more likely to be absent from work on days nearer the weekend. To test this he gathers the following data:

weekday	Mon	Tue	Wed	Thur	Fri
absentees	25	17	14	18	26

Carry out a χ^2 goodness of fit test to see if this data comes from a uniform distribution.

Do the test at the 5% level. Give a similar table of expected frequencies and comment on their values in terms of validity of the test. State the Null and Alternative hypotheses, the number of degrees of freedom, the p-value and the conclusion of the test, with a reason.

Also put the conclusion of the test into the context of the situation.

2 Yoghurts of a certain brand come in packs of six yoghurt tubs. There are different flavours. It is thought that the probability that a yoghurt tub is rhubarb flavour is $\frac{1}{6}$. To test this 60 packs were examined and the number of rhubarb-flavoured tubs X in each pack was recorded. The following results were obtained:

x	0	1	2	3	4	5	6
Observed frequency	7	17	20	10	5	1	0

a Under the Null hypothesis that this data comes from a $\mathrm{B}\left(6, \frac{1}{3}\right)$ distribution, construct a similar table for the expected frequencies.

b Explain why, from a validity point of view, this table, as it is, should not be used to carry out a ψ^2 goodness of fit test.

c Modify and combine the 2 tables, to give a table that is suitable for applying the test.

d Apply the test at the 5% level, state the number of degrees of freedom, the p-value and with a reason, the conclusion of the test.

Answers

1

Weekday	Mon	Tue	Wed	Thur	Fri
Expected absences	20	20	20	20	20

All the expected values are greater than 5, so there is not a validity problem

H_0 : this data come from a uniform distribution

H_1 : this data does not come from a uniform distribution

Degrees of freedom = 4 $p = 0.240$ (3 *sf*)

$0.240 > 0.05$ so we accept H_0

The employer does not have enough evidence to support his claim.

2 **a**

x	0	1	2	3	4	5	6
Expected frequency	5.27	15.8	19.8	13.2	4.94	0.988	0.0823

b there are expected values smaller than 5

c combine cells

x	0	1	2	3	≥ 4
Observed frequencies	7	17	20	10	6
Expected frequencies	5.27	15.8	19.8	13.2	6.01

d 4 degrees of freedom p-value = 0.838

$0.838 > 0.05$

We accept H_0 that this data comes from a $\mathrm{B}\left(6, \frac{1}{3}\right)$ distribution and that the probability that a yoghurt pot is rhubarb flavour is $\frac{1}{3}$

14.6 Choice, validity and interpretation of tests

1 The mass of the contents of a box of a type of cereal are known to be normally distributed with mean of μ grams and standard deviation of 5 grams. A quantity inspector wishes to test the null hypothesis that $\mu = 500$ against the alternative hypothesis that $\mu < 500$ at the 5% level. To do this she will measure the masses of a random sample of 100 cereal box contents and then calculate the sample mean $\bar{x}$

a Find the critical region for this test expressing the answer in terms of $\bar{x}$ and giving the answer to two decimal places.

b State the probability that the inspector will make a Type I error.

c If the value of μ is actually 498, find the probability that she will make a Type II error.

2 A box is known to contain either 2 red cards and one blue card or 2 blue cards and one red card. Only one card, unseen, can be taken out of the box at a time and then it must be replaced and the cards will be shuffled. In order to attempt to ascertain which cards are in the box, Sue devises the following test. She takes 5 cards from the box, one at a time, noting the colour and then replacing it in the box. She sets up the following hypotheses and decision rules:

H_0 : the box contains 2 red and 1 blue H_1 : the box contains 2 blue and 1 red.

If she obtains more red than blue cards she will accept H_0 and if she obtains more blue than red cards she will accept H_1.

a Find the probability that she makes a Type I error.

b Find the probability that she makes a Type II error.

Answers

1 **a** Under H_o , $\bar{X}$ ~ $N\left(500, \frac{5^2}{100}\right)$. Critical region given by $P\left(\bar{X} < A\right) = 0.05$

So the critical region is $\bar{x} < 499.18$ (2 *dp*)

b Probability of a Type I error is the level of the test which is 0.05

c Probability of a Type II error is

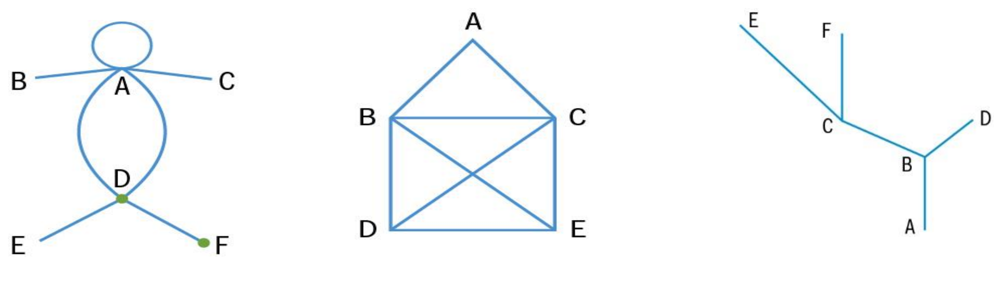
P(accepting $H_0 \mid H_1$ is true) = $P[\bar{X} > 499.18 \mid \bar{X} \sim N(498, 5^2/100)] = 0.00914$ (3 sf)

2 **a** $P(\text{Type I error}) = P(\text{reject } H_0 | H_0 \text{ is true}) = P(X = 0, 1, 2)$, where X ~ B(5, 2/3),
= 0.210 (3 *sf*)

b $P(\text{Type II error}) = P(\text{accept } H_0 | H_1 \text{ is true}) = 0.210$ (3 *sf*), $\frac{17}{81}$ by symmetry

15.1 Constructing graphs

1



(i) (ii) (iii)

Consider the 3 graphs above.

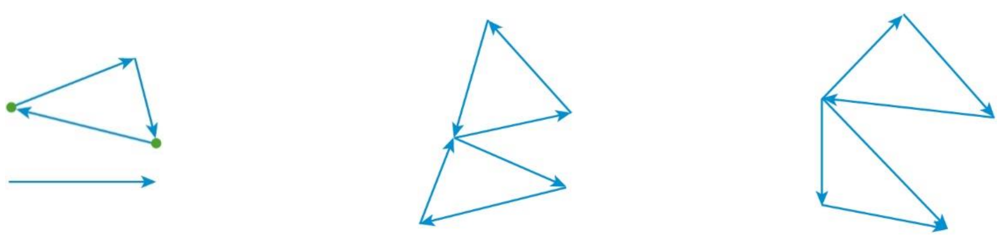
a For each graph state the degree of each vertex.

b For each graph verify that the hand-shaking lemma

"Sum of the degrees equals twice the number of edges",

is true.

2



(i) (ii) (iii)

Consider the three directed graphs given above.

a For each directed graph state if it is connected or disconnected.

b For those directed graphs that were connected, state whether or not they are strongly connected.

Answers

1 **a** **i** A6, B1, C1, D4, E1, F1

ii A2, B4, C4, D3, E3

iii A1, B3, C3, D1, E1, F1

b **i** 6+1+1+4+1+1=14=2 times 7 edges

ii 2+4+4+3+3=16=2 times 8 edges

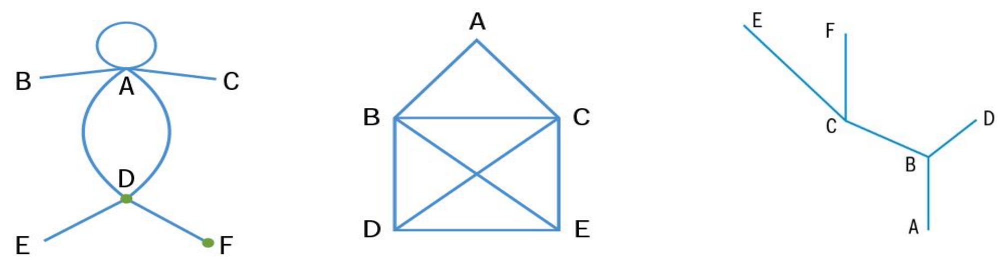
iii 1+3+3+1+1+1+1=10=2 times 5 edges

2 **a** Only (i) is disconnected

b (ii) is strongly connected, (iii) is not strongly connected.

15.2 Graph theory for unweighted graphs

1



(i) (ii) (iii)

Consider the 3 graphs above.

a State which graphs have a loop.

b State which graphs have multiple edges.

c State which graphs are simple.

2 **a** The matrix below is the adjacency matrix for an undirected graph. Draw the graph.

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

b State if this graph is simple or not. Name two features of the adjacency matrix that made you come to your conclusion.

3 **a** The matrix below is the adjacency matrix for a directed graph. Draw the graph.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b State if this graph is strongly connected or not. Name a feature of the adjacency matrix that made you come to your conclusion.

c In general for a directed graph state

i what the sum of the entries of the i^{th} row represents

ii what the sum of the entries of the i^{th} column represents.

4 For each of the following statements, give a reason why such a graph could not exist.

a A graph with a degree sequence of 1, 2, 3, 4, 5, 6, 8.

b A simple graph with a degree sequence of 1, 2, 3, 4, 4, 6

c A simple graph with a degree sequence of 0, 1, 2, 4, 4, 5

5 This question is about undirected graphs.

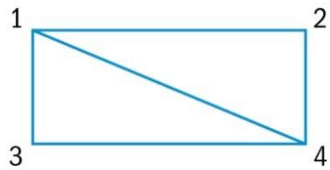
a Explain why the adjacency matrix will be symmetrical about the leading diagonal.

b State what the sum of the entries in the i^{th} row of the adjacency matrix will represent.

c State what all the entries on the leading diagonal being 0 would indicate.

d State what an entry larger than 1 would indicate.

e For the graph below write down the adjacency matrix.



f Find the number of walks of length 2 from vertex 1 to vertex 4. Write them down.

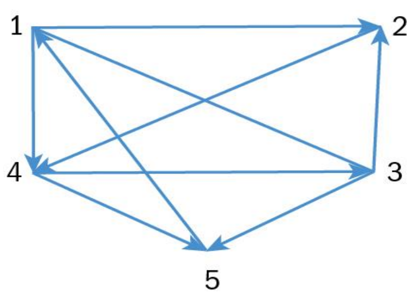
g Find the number of walks of length 3 from vertex 2 to vertex 3.

h Find the number of walks of length 4 from vertex 1 to vertex 1.

i Find the number of walks of length 10 from vertex 3 to vertex 4.

j Find the number of walks of length less than 5 from vertex 1 to vertex 4.

6 **a** For the directed graph below write down the adjacency matrix.



b Find the number of walks of length 2 from vertex 4 to vertex 1. Write them down.

c Find the number of walks of length 3 from vertex 4 to vertex 4.

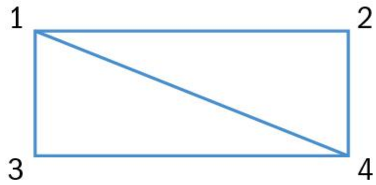
d Find the number of walks of length 4 from vertex 3 to vertex 4.

e Find the number of walks of length 10 from vertex 2 to vertex 3.

f Let the adjacency matrix be denoted by **A** . Verify that all the entries in matrix $\mathbf{A}^{10}$ are positive. State what this tells you about the directed graph.

g Find the length of the shortest walk from vertex 2 to vertex 1. Explain your answer with reference to powers of **A.**

7 **a** For the following undirected graph, find the transition matrix for a random walk round this graph.



b Find the steady state vector for this transition matrix.

c State what the sum of the entries in a column of any transition matrix will always add up to.

8 **a** The following adjacency matrix represents a directed graph. Draw this directed graph.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

b State if the graph is strongly connected or not.

c Find the transition matrix for a random walk round this graph.

d Find the steady state vector for this transition matrix.

Answers

1 **a** Only (i)

b Only (i)

c (ii) and (iii)

d **i** A6, B1, C1, D4, E1, F1

ii A2, B4, C4, D3, E3

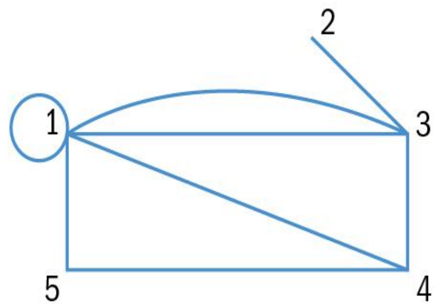
iii A1, B3, C3, D1, E1, F1

e **i** 6+1+1+4+1+1=14=2 times 7 edges

ii 2+4+4+3+3=16=2 times 8 edges

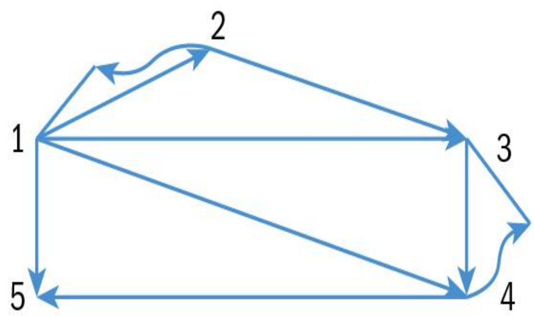
iii 1+3+3+1+1+1=10=2 times 5 edges

2 **a**



b Not simple, there is a 1 on the leading diagonal, a loop; there is an entry of 2, multiple edges.

3 **a**



b Not strongly connected; the bottom line is all 0, so no edges out from vertex 5

c **i** the out-degree of that vertex **ii** the in-degree of that vertex

4 **a** Sum of degrees is 29 which contradicts the hand-shaking lemma that says it should be twice the number of edges

b There are 6 vertices and so for a simple graph the maximum a degree can be is 5

c There are 6 vertices. The vertex of degree 5 is adjacent to all the other vertices but the vertex of degree 0 is not adjacent to any other vertex.

5 **a** There are the same number of edges from *i* to *j* as from *j* to *i*.

b It will give the degree of vertex *i*

c The absence of any loops

d The existence of multiple edges

e

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

f 2, 124 and 134

g 2

h 15

i 2929, all from considering the appropriate power of $\mathbf{A}$

j 21 from $\mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4$

6 **a**

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b 2, 451 and 431

c 3

d 1

e 15, all from considering the appropriate power of $\mathbf{A}$

f Verification of all positive shows that the directed graph is strongly connected.

g Shortest walk is of length 3, since entry, in row 2 column 1, is 0 in $\mathbf{A}$ and $\mathbf{A}^2$ but not in $\mathbf{A}^3$

7 **a**

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

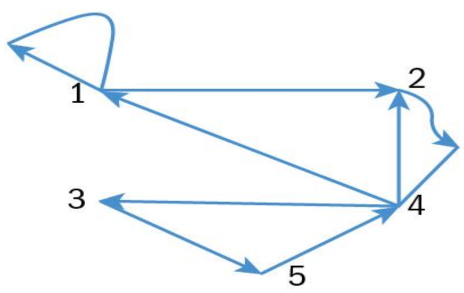
b

$$\begin{pmatrix} \frac{3}{10} \\ \frac{2}{10} \\ \frac{2}{10} \\ \frac{3}{10} \end{pmatrix}$$

Obtained either by solving 4 simultaneous equations or taking a very high power of the matrix.

c It will always add to 1

8 a



b It is strongly connected.

c

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

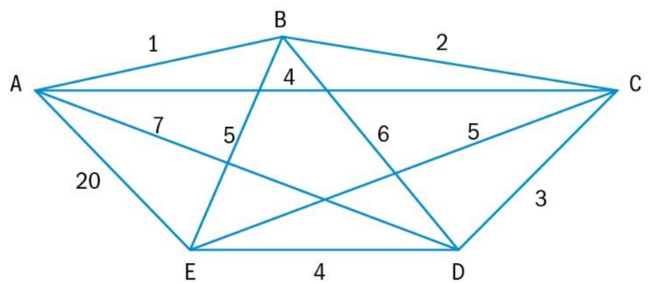
d

$$\begin{pmatrix} \frac{2}{9} \\ \frac{2}{9} \\ \frac{1}{9} \\ \frac{3}{9} \\ \frac{1}{9} \end{pmatrix}$$

Obtained either by solving 5 simultaneous equations or taking a very high power of the matrix.

15.3 Graph theory for weighted graphs: the minimum spanning tree

1 Consider the following weighted, complete graph.



Find a minimum spanning tree and state its total weight, stating the order in which the edges are added:

a Using Kruskal's algorithm

b Using Prim's algorithm starting at vertex C.

2 There is a security system in a large office block. Some computers are connected to others by very expensive cabling of different lengths. Taking the computers as the vertices and the cables as the edges, this system can be represented by a weighted, undirected graph. When it was installed the graph was connected. A very loud alarm will sound if and only if the graph becomes disconnected. A naughty but enterprising burglar decides to steal as much cabling as he can, without the alarm being set off. State what his best strategy would be if he wishes to maximise the length of cabling that he steals. You can assume that the burglar, although naughty, is intelligent and has studied graph theory.

Answers

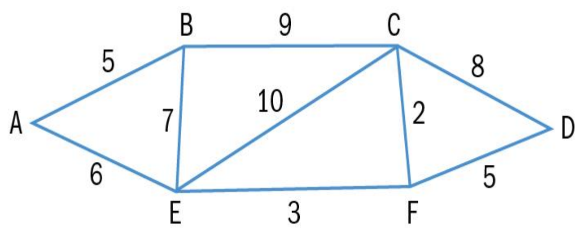
1 **a** AB(1) BC(2) CD(3) DE(4); total weight 10

b CB(2) BA(1) CD (3) DE(4); total weight 10

2 He should find a minimum spanning tree for the graph and then steal all the edges that are not in this tree.

15.4 Graph theory for weighted graphs: the Chinese postman problem

1 **a** State the condition for a graph to be Eulerian.



b For the weighted graph given above, state how many vertices of odd degree there are and state which ones they are.

c Find an Eulerian trail for this graph.

d Solve the Chinese Postman problem for this graph, giving the total distance that will have to be walked.

Answers

1 **a** All vertices must have even degree

b 2, B and F

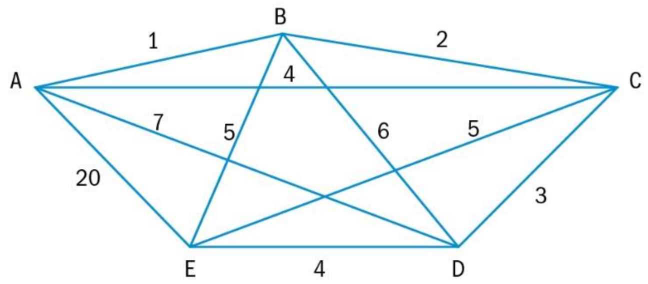
c For example BAEBCEFDCF

d Need the Eulerian trial from part (c) together with the shortest path from B to F which is BE(7)EF(3)

A solution is BAEBCEFDCFEB
Total length 5+6+7+9+10+3+5+8+2+7+3=65

15.5 Graph theory for weighted graphs: the travelling salesman problem

1 Consider the following weighted, complete graph



a Find an upper bound to the Travelling Salesman problem for this graph using the Nearest Neighbour algorithm,

i starting from vertex A,

ii starting from vertex E and first going to B.

b State which is the better of these two upper bounds.

c Find a lower bound to the Travelling Salesman problem for this graph using the Deleted Vertex algorithm,

i deleting vertex A,

ii deleting vertex E.

d State which is the better of these two lower bounds.

Answers

1 **a** **i** ABCDEA 1+2+3+4+20=30

ii EBACDE 5+1+4+3+4=17

b 17 is the better upper bound

c **i** Spanning tree for graph without A is BC(2)CD(3)ED(4)

Adding in 2 shortest edges into A AB(1)AC(4)

Gives lower bound as 14

ii Spanning tree for graph without E is AB(1)BC(2)CD(3)

Adding in 2 shortest edges into E ED(4)EB(5) (or EC5)

Gives lower bound as 15

d 15 is the better lower bound